CHAPTER 12

Velocity Vectors Acting on Objects

(B) Relative Velocity Vector

12.0 Prelude

Vectors, in physics, always act on an object.

Interaction: When the presence of an object affects the neighboring object(s) and vice versa, the said objects are said to interact with one another

12.1 Introducing the Concept of Relative Velocity

For the purposes of this textbook, relative velocity will, in general, be defined as a velocity which is the vector difference of two uniform velocities in the horizontal plane. Let the velocity of object A be called v_A and that of object B be called v_B. Let the velocity of A relative to the velocity of B be called v_{AB}. Then:

$$v_{AB} = v_A - v_B \qquad \text{..........(1)}$$

As a simple example, suppose you are travelling on a highway at 80 km/h and wish to pass a truck, travelling at 60 km/h, ahead of you, in the same direction. The speed with which you are approaching the truck will obviously be (80 - 60) = 20 km/h. If initially you were 500 m behind the truck then it is easy to calculate that it will take you 90 seconds to pass the truck The approach velocity (20 km/h) is the *relative velocity.*

Suppose the truck was travelling in a direction opposite to that of yours (on a parallel track) and you wanted to know how soon the two of you will cross each other. The approach velocity will now be (80 + 60) = 140 km/h. From an initial distance-apart of 500 m, it will take the two of you about 13 seconds to whiz by each other. Now, why did we *add* the velocities? Isn't the relative velocity supposed to be the difference of the two velocities? Well, in the reference frame of the car, its velocity is positive but that of the truck is negative because of its direction being opposite to that of yours. Subtracting a negative quantity, leads us to addition.

In the above two examples, all velocities were along the same straight line, so we could treat them as scalers. If things are that simple, we may not need a whole chapter on relative velocities. However, situations have a tendency to become ugly. Suppose the two velocities are not along the same straight line (so they could not be treated as scalers) but were at some unfriendly angle of (say) $\theta = 26.5794°$ in the second quadrant of the horizontal plane, then what do we do?

For situations of this type (and there are more where it came from), we need to hire an independent observer, unrelated to both objects. Such an observer will be an *observer-outside*. The *observer-outside* (by definition) must be at rest for both moving objects, (sitting on the pavement, shall we say) and, as such, is not going anywhere. The *observer-outside* (as always) uses a reference frame which by definition, is glued to the observer and as observer is glued to the pavement, the reference frame remains fixed in space for both moving objects. For this reason we call it a *neutral* reference frame. Let's write N for it. The velocity of object A is measured as v_{AN} by the hire, and that of the object B, as v_{BN}. The *resultant velocity* of object A with respect to that of object B is v_{AB} .

12.2 Mathematical Representation of "Relative Velocity"

The relative velocity v_{AB} is defined as the difference of v_{AN} and v_{BN}. We write:

$$v_{AB} = v_{AN} - v_{BN} \qquad \text{..........(2)}$$

As there is no subtraction in vectors, we must construct the negative vector of vector v_{BN}. This is done by switching the tip and the toe. In the present situation, however, we switch the subscripts around to achieve the same effect:

$$-v_{BN} = v_{NB} \qquad \text{..........(3)}$$

Eqn (2) can be written as;

$$v_{AB} = v_{AN} + (-v_{BN})$$

Using Eqn (3) we get:

$$v_{AB} = v_{AN} + v_{NB} \qquad \text{..........(4)}$$

Eqn (4) is the all-important *relative velocity* equation.

In words, the equation tells us that the velocity of object A relative to the velocity of object B equals the vector combination of the velocity of object A as measured by the hire, v_{AN}, and that of the hire as measured by object B i.e. v_{NB}.

12.3 The Order of the Subscripts

The order of subscripts, as they appear in Eqn (4), is very important. The left subscript (subscript A) on the left hand side of the equation, is on the extreme left on the right hand side of the equation. The right subscript (subscript B) on the left hand side of the equation, is on the extreme right on the right hand side of the equation. The two N's (representing the neutral reference frame), in the two terms on the right hand side of Eqn (4) are facing each other. This configuration of subscripts is of extreme importance. We must check this order of subscripts for all relative velocity equations that we develop, before we use them.

12.4 Two Properties of the Relative Velocity Equation

(1) Permutation

The relative velocity equation, Eqn (4), can be *permutated* for any of the three velocities for any pair of subscripts. As long as we maintain the order of the subscripts (as described above), the equations will be fully valid and meaningful. As an example, permutate Eqn (4):

$$v_{AB} = v_{AN} + v_{NB} \qquad v_{BA} = v_{BN} + v_{NA} \qquad v_{AN} = v_{AB} + v_{BN}$$

$$v_{NA} = v_{NB} + v_{BA} \qquad v_{BN} = v_{BA} + v_{AN} \qquad v_{NB} = v_{NA} + v_{AB}$$

Interpreting the Permutations:

(1) A boat is crossing a river while fighting the water current to stay on course. We want to know the velocity of the *boat* with respect to the *shore*.

(2) A boat is crossing a river while fighting the water current to stay on course. We want to know the velocity of the *shore* with respect to the *boat*.

(3) A boat is crossing a river while fighting the water current to stay on course. We want to know the velocity of the *boat* with respect to the *water current*.

(4) A boat is crossing a river while fighting the water current to stay on course. We want to know the velocity of the *water current* with respect to the *boat*.

(5) A boat is crossing a river while fighting the water current to stay on course. We want to know the velocity of the *water current* with respect to the *shore*.

(6) A boat is crossing a river while fighting the water current to stay on course. We want to know the velocity of the *shore* with respect to the *water current.*

Table 1: Explanations

First Object and its Velocity	Second Object and its velocity	We Want to Know (relative velocity)	Relative velocity Equation
Boat Velocity of boat with respect to still water: v_{BW}	*Water* Velocity of water current with respect to the shore: v_{WSh}	*Boat* Velocity of boat with respect to the shore: v_{BSh}	$v_{BSh} = v_{BW} + v_{WSh}$
Shore Velocity of shore with respect to the water current: v_{ShW}	*Water* Velocity of water current with respect to the boat: v_{WB}	*Shore* Velocity of shore with respect to the boat: v_{ShB}	$v_{ShB} = v_{ShW} + v_{WB}$
Water CUrrent Velocity of water current with respect to the shore: v_{WSh}	*Shore* velocity of shore with respect to the boat: v_{ShB}	*Water Current* velocity of water current with respect to the boat: v_{WB}	$v_{WB} = v_{WSh} + v_{ShB}$
Boat Velocity of boat with respect to the shore: v_{BSh}	*Shore* Velocity of shore with respect to the water current: v_{ShW}	*Boat* velocity of boat with respect to the water current: v_{BW}	$v_{BW} = v_{BSh} + v_{ShW}$
Shore Velocity of shore with respect to the boat: v_{ShB}	*Boat* velocity of boat with respect to the water current: v_{BW}	*Shore* velocity of shore with respect to the water current: v_{ShW}	$v_{ShW} = v_{ShB} + v_{BW}$
Water Current Velocity of water current with respect to the boat: v_{WB}	*Boat* velocity of boat with respect to the shore: v_{BSh}	*Water Current* velocity of water current with respect to the shore: v_{WSh}	$v_{WSh} = v_{WB} + v_{BSh}$

(2) Extension

The relative velocity equation can also be extended if we run into more than one neighboring objects moving with their own independent velocities in their own independent directions. Consider the following equations with two (R and S) and three (R, S and T) neutral reference objects:

$$v_{AB} = v_{AR} + v_{RS} + v_{SB} \qquad ....(5)$$

$$v_{AB} = v_{AR} + v_{RS} + v_{ST} + v_{TB} \qquad ....(6)$$

Interpreting Equations 4, 5 and 6:

Eqn (4): A boat is crossing a river while fighting the water current to stay on course. We want to know the velocity of the boat with respect to the shore.

Eqn (5): A passenger is walking across the deck of the boat which is crossing the river, fighting the water current to stay on course. We want to know the velocity of the passenger with respect to the shore.

Eqn (6): A bug is walking on the back of the passenger who is walking across the deck of the boat which is crossing the river, fighting the water current to stay on course. We want to know the velocity of the bug with respect to the shore.

Table 2: Examples

Our Object and its Velocity	Neighboring Object(s) and its velocity	Relative velocity	Relative velocity Equation
Boat velocity of the boat with respect to the still water v_{BW}	*(1) Water Current* velocity of the water current with respect to the shore: v_{WSh}	*Boat* velocity of the boat with respect to the shore: v_{BSh}	$v_{BSh} = v_{BW} + v_{WSh}$
Passenger velocity of the passenger with respect to the boat: v_{PB}	*(1) Boat* velocity of the boat with respect to the still water: v_{BW} *(2) Water Current* velocity of the water current with respect to the shore: v_{WSh}	*Passenger* velocity of the passenger with respect to the shore: v_{PSh}	$v_{PSh} = v_{PB} + v_{BW} + v_{WSh}$
Bug velocity of the bug with respect to the passenger: v_{BugP}	*(1) Passenger* velocity of the passenger with respect to the boat: v_{PB} *(2) Boat* velocity of the boat with respect to the still water: v_{BW} *(3) Water Current* velocity of the water current with respect to the shore: v_{WSh}	Bug velocity of the bug with respect to shore: v_{BugSh}	$v_{BugSh} = v_{BugP} + v_{PB} + v_{BW} + v_{WSh}$

12.5 For Your Information:

(1) Even though objects and their neighbors are free to travel either with uniform velocities or with non-uniform velocities, we shall be restricting ourselves to uniform velocities only. The equation of motion that we shall use, is

$$v = d/t \qquad \text{..........(7)}$$

(2) We shall be given only one neighboring object, in one problem. (I hope!)

(3) Velocity of a ship or boat is always measured with respect to still water (and not with respect to the shore).

(4) Velocities of airplanes are always measured with respect to still air (and not with respect to the ground below). Even for take-off (when the airplane is on the runway), the take-off velocity is measured with respect to air and not with respect to runway.

(5) Wind directions are always given in terms of the direction from where they are coming; and not in terms of the direction in which they are going. They always say *wind is blowing from north-west* or will simply refer to a wind as *north-westerly* wind. The vector representing the direction of this wind will point toward south east!

(6) Water in a river or canal or stream always flows downstream.

12.6 WARNING:

Relative Velocity Equation is a Vector Equation

Please be advised that Eqn (4) is a vector equation. It tells us that the left hand side vector, v_{AB} , is the resultant vector of the two right hand side vectors v_{AN} and v_{NB} . Vectors v_{AN} and v_{NB} must be combined vectorially (i.e. tiptoed).

12.7 Solving the "Relative Velocity" Equation

(i) Vectors are Collinear

If the two velocities v_{AN} and v_{NB} are parallel or anti-parallel, rules #1 and #2 of vector combination will be applicable. We shall, however, first construct the negative vector of the appropriate vector (changing it from parallel to anti-parallel or vice versa) and then treat all vectors as scaler entities.:

(ii) Vectors are Not Collinear

If the three vectors are not collinear, they will form a triangle. We shall first tiptoe the two vectors that are on the right hand side of the equation, and then complete the triangle by using the vector on the left hand side of the equation, as the third side of the triangle. Thus vectors v_{AN} and v_{NB} must be *tiptoed* (to form two sides of the triangle) and then v_{AB} must be drawn as the resultant vector to complete the triangle. As vectors always come with their orientations, v_{AN} and v_{NB} will be found to be at some angle θ with respect to one another. Forming a triangle, therefore, will never be a problem. We write:

The purpose of the relative velocity equation is to form a triangle.

After the triangle has been constructed, the original equation becomes redundant. It should be totally ignored *and treated as if it never existed.*

12.8 Format for Solving the Relative Velocity Equation

(i) Identify the *A* object and the *B* object. The object called the *A* object is the one whose speed is *to be found,* and as such is the center of attention. The other object is the *other* object. Also note down which of the three velocities is to be determined.

(ii) Write the relative velocity equation (in the form of Eqn 4) for the *to be determined* velocity. Check the order of subscripts for correctness.

(iii) Construct the necessary negative vector. In some cases, the given velocities may fit in the equation directly and we may not have to construct a negative vector

(iv) Form a triangle (except when the vectors are collinear), by tiptoeing the two vectors on the right hand side of eqn (4). Even though the order in which the vectors can be combined is immaterial, ***always choose the longer vector first!***

(v) Forget the relative velocity equation (Eqn 4), and treat it as if it never existed

(vi) Use sine law or the cosine law or both to solve the problem

12.9 The Role of "Time"

Time of interaction is the duration of time for which the two objects are affecting each other. There is thus, only one value of *time,* which can be calculated from any of the three sides of the triangle, whichever one is most economical or is the *cheapest*. One would of course, use Eqn (7).

12.10 The Role of Displacement

Let's change the velocities in Eqn (4) into corresponding displacements (and time) by using Eqn (7). We get:

$$\frac{d_{AB}}{t} = \frac{d_{AR}}{t} + \frac{d_{RB}}{t}$$

Cancelling out t, from both sides, we get:

$$d_{AB} = d_{AR} + d_{RB} \qquad ....(8)$$

Eqn (8) is also a vector equation. We shall tiptoe vectors d_{AR} and d_{RB} to form two sides of a triangle and then complete the triangle with vector d_{AB}. *Sine* and *cosine* laws will be applicable. This triangle may prove to be quite useful in solving some problems.

It is of interest to point out that the velocity-triangle and the displacement-triangles will necessarily be *similar* to one another. The corresponding angles of the two triangles will be identical. From the properties of *similar* triangles, we notice that the ratios of corresponding sides (of the velocity-triangle and the displacement-triangle) will be equal. Written in the equation form, we get:

$$\frac{v_{AB}}{d_{AB}} = \frac{v_{AR}}{d_{AR}} = \frac{v_{RB}}{d_{RB}} \quad or \quad \frac{d_{AB}}{v_{AB}} = \frac{d_{AR}}{v_{AR}} = \frac{d_{RB}}{v_{RB}} \qquad ....(9)$$

12.11 Are We Ready Now?

Yes.

12.12 We shall solve some problems as examples:

Example (1)

Data A car, travelling at 90 km/h, passes a truck travelling at 60 km/h.

(a) Find The velocity of the car with respect to the truck.

Solution object A is the car and object B is the truck. Let road R be the neutral reference frame. The velocities v_{AR} and v_{RB}, as measured by the *observer outside* are:,

$$v_{AR} = +90 \text{ km/h, (east)} \quad , \& \quad v_{BR} = +60 \text{ km/h, (east)}$$

The relative velocity equation is:

$$v_{AB} = v_{AR} + v_{RB}$$

Construct the negative vector of the object B's velocity vector, v_{BR}, to get v_{RB} :

$$v_{RB} = -60 \text{ km/h (west)}$$

As the vectors are collinear, we shall solve the equation scalerly;

$$v_{AB} = 90 + (-60) = \mathbf{30\ km/h} \qquad \textit{Answer}$$

(b) Find If the car passed the truck in 3 sec., for how many meters did the two vehicles travel side-by-side during the activity time?

Solution $30\text{km/h} = 30 \div 3.6 = 8.3333 \text{ m/s}$

$$v = \frac{d}{t} \qquad 8.3333 = \frac{d}{3} \qquad \mathbf{d = 25.000\ m} \qquad \textit{Answer}$$

Please note that the car travelled 75 m and the truck travelled 25.

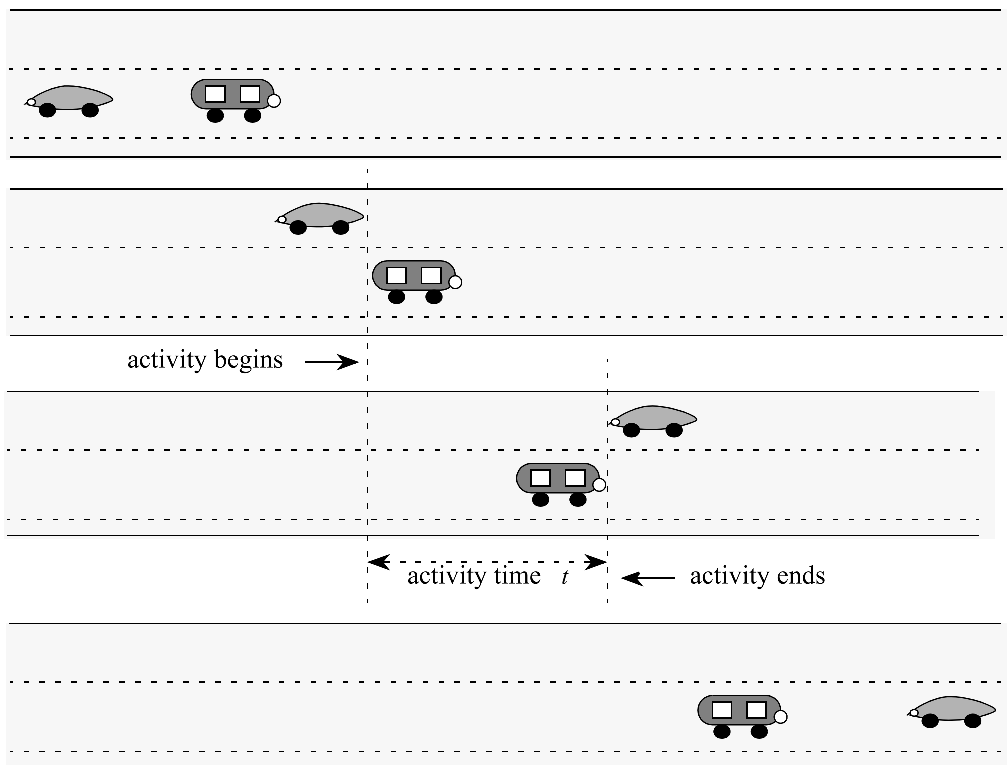


Fig (1) The Mechanics of "Passing" on a Highway.

Example (2)

Data: A car is travelling north at 90 km/h. A truck is travelling south, in the adjoining lane, at 60 km/h.

(a) Find The relative velocity of the car with respect to the truck.

Solution We rotate our reference frame so that the north-south direction becomes our x-mode. The two vehicles, as determined by the *observer-outside* are:

$v_{AR} = +90$ km/h, (north) & $v_{BR} = -60$ km/h, (south)

The relative velocity equation is:

$$v_{AB} = v_{AR} + v_{RB}$$

Construct the negative vector of the neighbor's velocity vector, v_{BR}, to get v_{RB} :

$$v_{RB} = +60 \text{ km/h (north)}$$

As the vectors are collinear, we shall solve the equation scalerly;

$$v_{AB} = 90 + (+60) = \mathbf{150\ km/h}$$ ***Answer***

(b) Find: If the two vehicles traversed a distance of 13 meters, how long it took them to pass one another?

Solution: $150\text{km/h} = 150 \div 3.6 = 41.6667 \text{ m/s}$

$$v = \frac{d}{t} \qquad 41.6667 = \frac{13}{t} \qquad t = \mathbf{0.312\ sec}$$ ***Answer***

Please note that out of 13 m (the total overlap distance), the car covered a distance of 7.8 m and the truck travelled for 5.2 m.

Example (3)

Data A car is travelling eastward at 90 km/h. A truck is travelling due north at 60 km/h

(a) Find: The velocity of the car with respect to the truck.

Solution We let our reference frame coincide with the east direction. North will then be along the y-axis of our reference frame. The two vehicles determined by the *observer-outside* are.

$v_{AR} = +90$ km/h due east & $v_{BR} = +60$ km/h due north

The relative velocity equation is:

$$v_{AB} = v_{AR} + v_{RB}$$

Construct the negative vector of the neighbor's velocity vector, v_{BR}, to get v_{RB} :

$$v_{RB} = +60 \text{ km/h (south)}$$

As the vectors are not collinear, we must form a triangle.

$$v_{AB} = \sqrt{90^2 + 60^2} = \mathbf{108.167\ km/h}$$

$$\theta = tan^{-1}\left(\frac{60}{90}\right) = \mathbf{33.690°\ south\ of\ east}$$

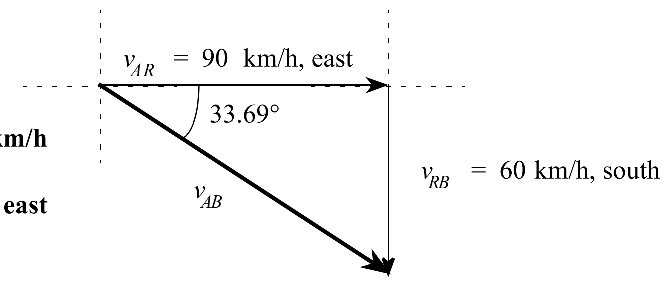


Fig (1) Relative Velocity of the Car and the Truck

The triangle in this case is found to be a right angled triangle. So we got away using the simpler equation, the Pythagorean theorem.

108.167 km/h at 33.690° south of east ***Answer***

(b) Find: If the two vehicles crossed the origin of a reference frame at $t = 0$, (one after the other, in quick succession), how far apart will the two be after one minute?

Solution $108.167 \text{ km/h} = 108.167 \div 3.6 = 30.0464 \text{ m/s}$

$v = d/t$ $30.0704 = d/60$ $\mathbf{d = 1802.776\ m}$ ***Answer***

Please note that the car would have travelled 1500 m along the positive x-axis while the truck would have travelled 1000 m in the northerly direction.

Example (4)

Data A boat wishes to cross a river and heads to a point directly opposite the starting point. The speed of the boat relative to still water v_{BW}, is 3.0 m/s. Water is rushing down the river with a velocity relative to shore of v_{WSh} 1.20 m/s.

(a) Find: The velocity of the boat relative to shore v_{BSh}.

Solution The relative velocity equation for v_{BSh} can be written as:

$$v_{BSh} = v_{BW} + v_{WSh}$$

As the given velocities fit into the relative velocity equation just as they are, we do not need to construct any negative vector. The triangle is formed by tiptoeing the vectors v_{BW} and v_{WSh}.

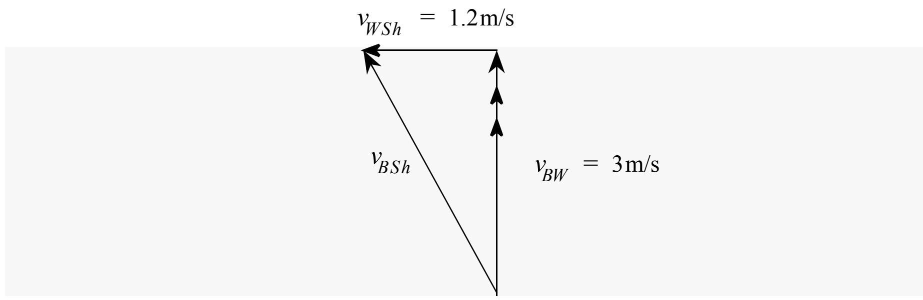


Fig (2) The Triangle of Velocities

The triangle turns out to be a right angled triangle. So we use the pythagorean theorem to find $v_{b,sh}$

$$v_{BSh} = \sqrt{3.0^2 + 1.20^2} = \mathbf{3.231\ \ m/s} \qquad \textbf{\textit{Answer}}$$

(b) Find: If the river is 600 m wide, how long does it take the boat to cross the river?

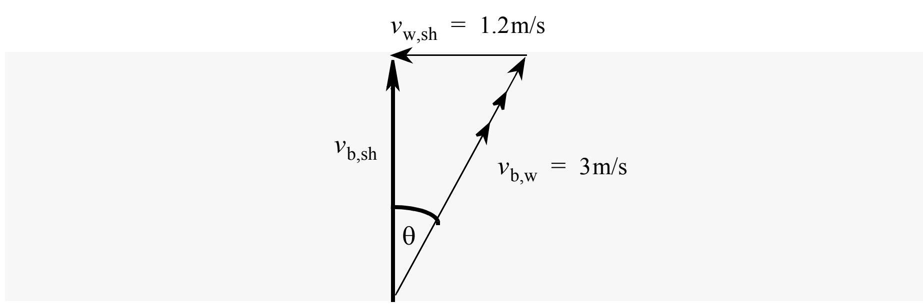
Solution We are looking for the activity time. It can be found from any of the three sides of the triangle by using the equation $v = d/t$. Thus:

$$3.0 = 600/t \qquad \boldsymbol{t} = \mathbf{200\ sec} \qquad \mathbf{3\ minutes\ and\ 20\ seconds} \qquad \textbf{\textit{Answer}}$$

(c) Find: How far down stream (from a point directly opposite its starting point) did the boat land on the opposite bank?

Solution The boat got pushed by the water for a time equal to the activity time: 200 sec. Using the equation $v = d/t$, we find:

$$1.20 = \frac{d}{200} \qquad \boldsymbol{d} = \mathbf{240\ m} \qquad \textbf{\textit{Answer}}$$



(d) Find: If the boat wanted to reach a point directly opposite its starting point, at what upstream angle θ should it head?

Solution From the right angled triangle, we find that

$$sin\theta = \frac{v_{w,sh}}{v_{b,w}} = \frac{1.2}{3} \qquad \theta = \mathbf{23.578°} \qquad \textbf{\textit{Answer}}$$

(e) Find: How long does the boat take to reach the other bank?

Solution We shall use the equation $v = d/t$. But if v is known, d is not known and vice versa. Let's find the distance travelled by water:

$$d_W/600 = tan 23.578 \qquad d_W = 261.859 \text{ m}$$

$$v = d/t \qquad 1.20 = 261.859/t \qquad \mathbf{t = 218.216 \text{ sec}}$$ ***Answer***

(f) Find What distance did the boat actually travel in order to get to a point directly opposite the starting position?

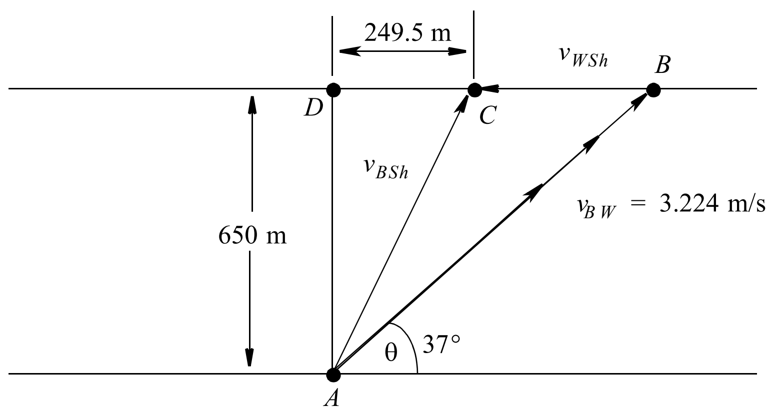
Solution Using the equation $v = d/t$, we find:

$$3.0 = d/218.216 \qquad \mathbf{d = 654.648 \text{ m}}$$ ***Answer***

It should be noted that the actual distance was only 600 m. The boat, however travelled 655 m to cover the 600 m long distance!

Example (5)

Data: See the diagram. Boat whose speed relative to still water $v_{B\,W} = 3.224$ m/s, starts from "A", travels toward "B" and reaches "C".



(a) Find Velocity of water relative to shore v_{WSh}

Solution
$$\angle ACD = tan^{-1}\left(\frac{650}{249.5}\right) = 69°$$

$$\angle ACB = 180 - 69 = 111°$$

$$\angle CAB = 180 - 111 - 37 = 32°$$

Using sine law:

$$\frac{3.224}{sin 111} = \frac{v_{B\,W}}{sin 32} \qquad \mathbf{v_{B\,W} = 1.83 \text{ m/s}}$$ ***Answer***

(b) Find: the time taken by the boat to reach "C".

Solution
$$\frac{AD}{AB} = \frac{650}{AB} = cos 53° \qquad AB = \frac{650}{cos 53°} = 1080 \text{ m}$$

$$v = \frac{d}{t} \qquad 3.224 = \frac{1080}{t} \qquad \mathbf{t = 334.988 \approx 335 \text{ sec}}$$ ***Answer***

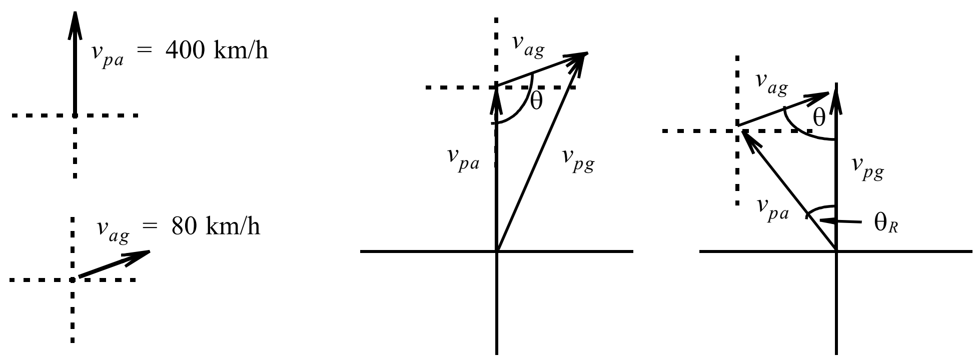
Example #6

Data A n airplane whose air speed is 400 km/h is to travel due north to its destination. A rather strong wind begins to blow at 80 km/h from south of west at $\angle\theta = 20°$.

(a) Find If the plane does not take any corrective measures, what will its velocity be relative to ground v_{pg}?

Solution The equation: $v_{pg} = v_{pa} + v_{ag}$.

Velocity of air with respect to ground will be 80 km/h at $\angle\theta = 20°$ directed north of east. The two vectors on the right hand side of the equation can be tiptoed directly. The angle needed for using the cosine law is $\angle\theta = 90 + 20 = 110°$



vectors are not drawn to scale (a) (b)

The answer v_{pg} will be found by using the cosine law. Now 400 km/h equals $400/3.6 = 111.111$ m/s, and 80 km/h equals $80/3.6 = 22.222$ m/s. Using cosine law, we get:

$$v_{pg} = \sqrt{(111.111^2 + 2.222^2 - 2 \times 111.111 \times 22.222 \times \cos 110)} = \mathbf{120.534 m/s}$$ ***Answer***

(b) Find If the plane must reach its destination (i.e. must keep travelling North), then in what direction should it fly?

Solution In this case v_{pg} is directed north. The magnitude of v_{pa} will remain 400 km/h. The triangle is shown in Fig (b). Also note that the $\angle\theta$ (opposite side v_{pa} is $\angle\theta = 90 - 20 = 70°$. We need to find $\angle\theta_R$. Using sine law, we get:

$$\frac{\sin\theta}{v_{pa}} = \frac{\sin\theta_R}{v_{ag}} \quad or \quad \frac{\sin 70}{400} = \frac{\sin\theta_R}{80}$$

giving us $\boldsymbol{\theta_R = 10.833°}$ west of north ***Answer***

Note: For this part we did not convert velocities to m/s.Can you guess why?

(c) Find the magnitude of v_{pg}.

Solution The angle opposite the side v_{pg} is $180 - 70 - 10.833 = 99.167°$. Using sine law:

$$\frac{v_{pg}}{\sin 99.167} = \frac{120.534}{\sin 70} \quad or \quad \mathbf{v_{pg} = 126.631\ m/s}$$ ***Answer***