CHAPTER 11

Velocity Vectors Acting on Objects
(A) Projectile Motion

11.0 Prelude

Vectors in physics, always act on an object.

11.1 Introduction

In this chapter we shall consider a case where two *velocity vectors* act on an object. We are referring here to *Projectile Motion*. The motion is parabolic in nature and is executed by objects, when they are thrown or kicked or fired in air. We call such objects *Projectiles*. We come across such a motion scores of time every day. Balls in almost all ball games, bullets, cannon shells, balls rolling off a table, fireworks exploding half way up in the sky, stones thrown up or down, all are projectiles. We call the path of motion of a projectile, a *trajectory*. A typical trajectory is shown below:

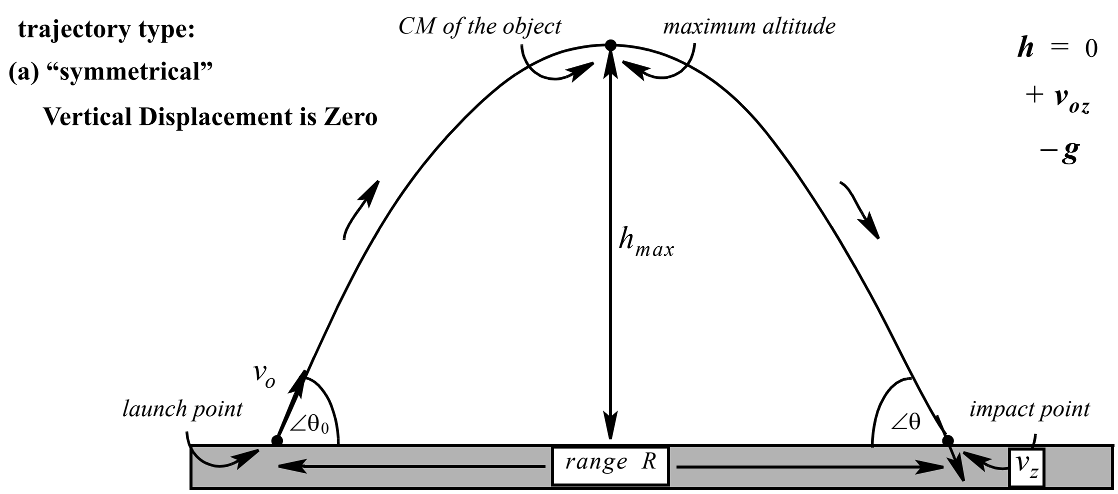

Fig (1) A Commonly Observed Trajectory

Additionally, we define the time that the projectile spends in air, as *time of flight* t_{fl}, the total horizontal distance traversed by the projectile as *range* R, and the maximum height reached by the object as the *maximum height* h_{max}. Fig (2) shows R and h_{max}.

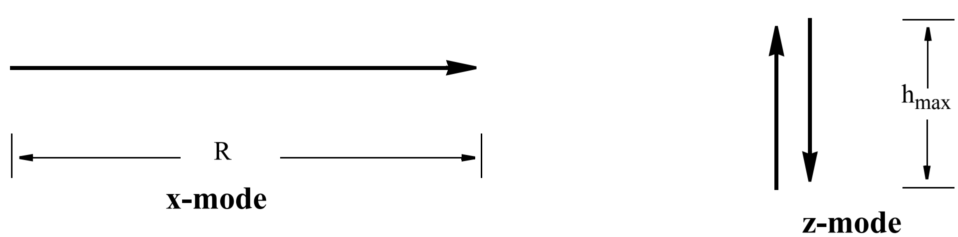

Fig (2) The Two Modes of the Projectile's Motion: (i) Displacements in x- and z-modes

11.2 Anatomy

The range R of the projectile is indicative of the presence of an x-mode velocity, while the (up-down motion) rising to maximum height and then coming back down, is indicative of the presence of a z-mode velocity. *Two velocity vectors, therefore, are acting simultaneously on the projectile.*

Once in air, the projectile is free of the (unglued) force that launched it. We would, therefore, expect the two velocities to be uniform. The x-mode velocity does indeed remain constant for the entire duration of the flight of the projectile. The z-mode velocity, on the other hand, simply cannot remain constant for reasons of earth's force of pull. It is therefore, an accelerated motion with acceleration being the acceleration due to gravity whose magnitude is 9.80 m/s². Fig (3) shows the x- and z-mode velocity-vectors plotted against time.

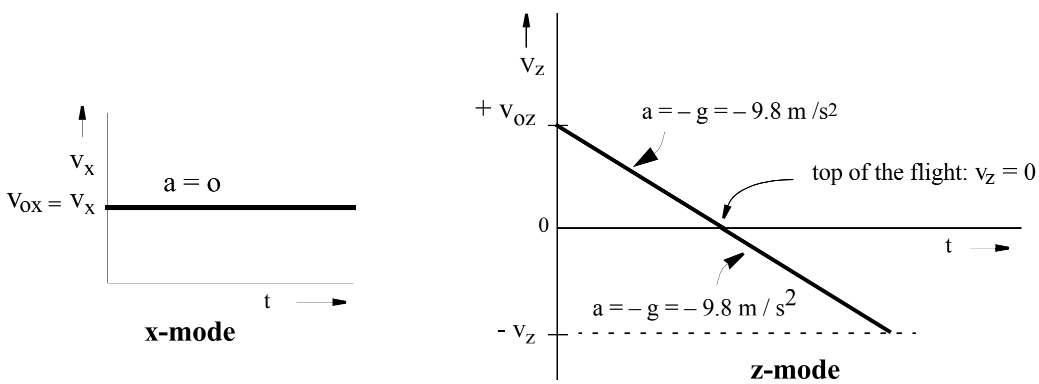

Fig (3) The Two Modes of the Projectile's Motion: (ii) Velocities in x- and z-modes

11.3 Parameters of Pro8jectile Motion

Following are the parameters of motion:

(i) initial velocity: the launch speed v_o and the launch angle θ_o (above or below the horizontal),

(ii) time of flight t_{fl} ,

(iii) maximum height h_{max} attained by the projectile (only if it was projected at some angle above the horizontal)

(iv) total horizontal distance traversed by the projectile: the range R

(v) impact velocity v and the impact angle θ

11.4 Types of Trajectories

The z-mode of a projectile's motion is *free fall*, as described in Chapter 6. As stated there, z-mode motion is of four types. Projectile trajectories, therefore, are also of four types. The first of these is shown in Fig (1), above. The other three are shown in Fig (4), (5) and (6). Algebraic signs of z-mode (i) displacement, (ii) velocity, and (iii) acceleration vectors are also shown. The axes used, show respectively the x-mode and the z-mode of the projectile's trajectory. Origin of the reference frame is placed at the starting position of the projectile. Displacement in x-mode R, and that in z-mode h, are measured with respect to this origin.

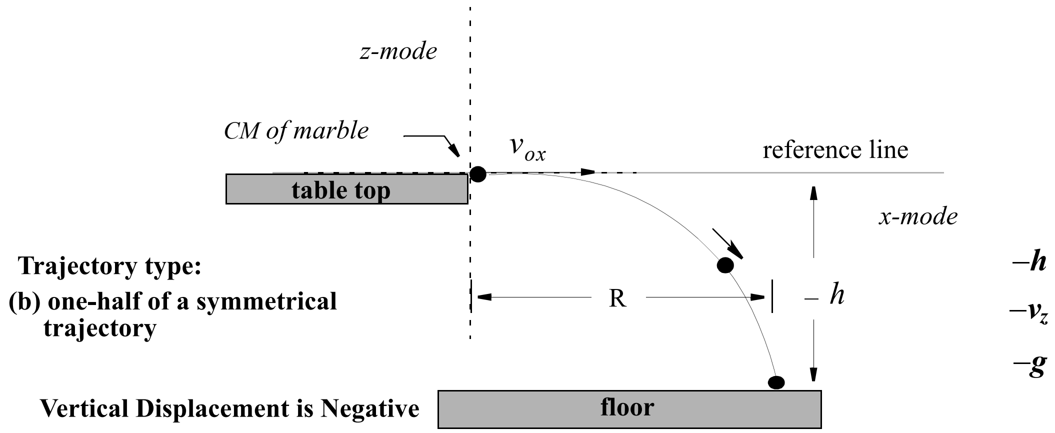

Fig (4) Trajectory with $h_{up} = 0$

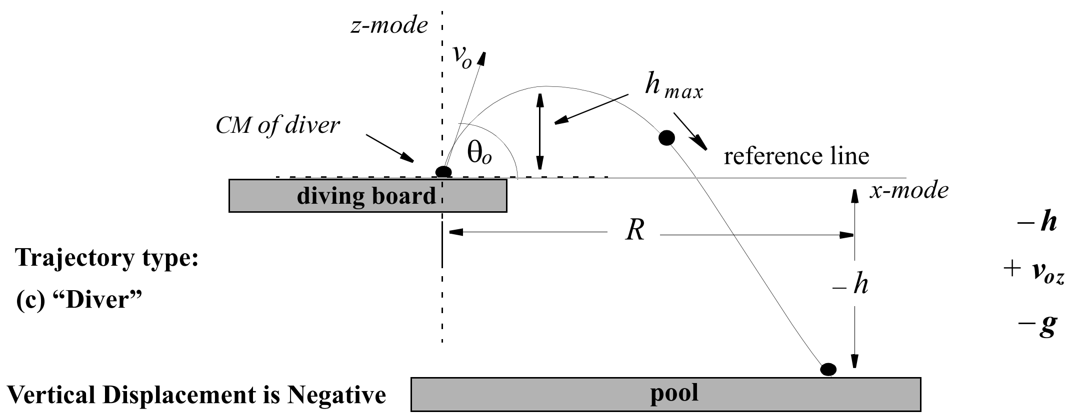

Fig (5) Trajectory with $h_{up} < h_{down}$

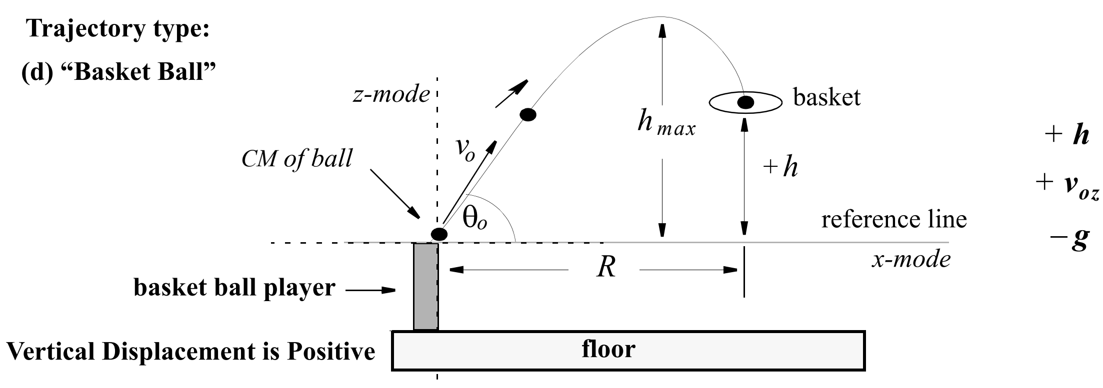

Fig (6) Trajectory with $h_{up} > h_{down}$

11.5 The Tools

(i) x-mode: As the velocity is uniform, we have only one equation:

$$v_{ox} = v_x = d/t \qquad \text{..........(1)}$$

(ii) z-mode: Motion in this mode is accelerated. Hence the entire set of z-mode kinematic equations may be required. These are reproduced here, for ready reference:

$$v_z = v_{oz} + gt \qquad v_z = gt \qquad \text{..........(2)}$$

$$h = v_{oz}t + (1/2)gt^2 \qquad h = (1/2)gt^2 \qquad \text{..........(3)}$$

$$h = (1/2)(v_{oz} + v_z)t \qquad h = (1/2)(v_{oz}t) \qquad \text{..........(4)}$$

$$h = (v_z^2 - v_{oz}^2)/(2g) \qquad h = (v_z^2)/(2g) \qquad \text{..........(5)}$$

11.6 The Analysis

The purpose of analysis, as always, is to determine one equilibrium state provided that the other is given. The parameters that we can be required to find, are one or more of t_{fl}, R, h_{max}, v_{impact} or v_{oz} etc.

The object is acted upon by two independent velocity-vectors that are in independent modes and thus are totally unconnected. The object, however, accomplishes the motion in time t_{fl}. The activity time, therefore, for each velocity-vector must also be t_{fl}. If we determine t_{fl} in one mode then this must also be valid for the other mode.

The identicity of t_{fl} in the two otherwise independent modes, is our key to solving problems. Find the devil (the t_{fl}) in one mode (any which one) and apply it to the other mode. With this conspirational strategy in mind, we design two bins: one each for the two modes and list all possible parameter for each mode.

Table 1: The Filled Bins,

x-mode	z-mode
$v_{ox} = v_x = d/t$	$h = v_{oz}t + (1/2)gt^2$
$v_{ox} = v_x =$ $R =$ $d = x =$ $t =$ $t_{fl} =$	$v_{oz} =$ $v_z =$ $h =$ $h_{max} =$ $t =$ $t_{fl} =$ $g = -9.8 \ (m/s^2)$

Now proceed as follows

(i) Read the problem carefully and dump data in the two bins of Table 1.

(ii) Find t_{fl}. Use the bin that permits you to do so.

(iii) Carry this t_{fl} to the other bin. By this time the solution of the problem will be quite obvious and you will not need any more prompting.

The analysis is complete and the problem is solved

11.7 "Mission Accomplished"? - Not So Fast!

We have some unfinished business to take care of. For example what if time of flight was not involved in our investigation at all? In this case finding t_{fl} will be a waste of time and will violate the *miss* principle. Again how do we know that the trajectory is a parabola? We may, on the side, develop some more useful equations for suitable trajectories. We shall begin by developing a time-independent equation and proving that the trajectory is indeed a parabola.

11-8 The Time Independent Equation
Trajectory of a Projectile is a Parabola

Using the above format, let us find t from the data in the x-bin. The time t that the projectile would take to traverse some distance x with its x-mode speed v_{ox} is:

$$v_{ox} = x/t \qquad t = x/v_{ox} \qquad \text{..........(6)}$$

We carry this value of t to the z-bin and plug it into that bin's equation. Remembering that g is always negative, we get.

$$h = (v_{oz})\left(\frac{x}{v_{ox}}\right) + (1/2)(-g)\left(\frac{x}{v_{ox}}\right)^2$$

$$h = \left(\frac{v_{oz}}{v_{ox}}\right)x - \left\{\frac{(g/2)}{(v_{ox})^2}\right\}x^2 \qquad \text{..........(7)}$$

But $\quad v_{oz}/v_{ox} = tan\,\theta_o,$

$$h = (tan\,\theta_o)\,x - \left\{\frac{(g/2)}{(v_{ox})^2}\right\}x^2 \qquad \text{..........(8)}$$

Eqn (8) is our time-independent equation. It correlates the z-mode displacement $\boldsymbol{h}$ and the x-mode displacement $\boldsymbol{x}$, without going through the *time* channel. For aa given $\boldsymbol{h}$, we can find $\boldsymbol{x}$ and vice versa.

For our good fortune, Eqn (8) is also comparable to the equation of a parabola:

$$y = ax - bx^2 \qquad \text{..........(9)}$$

where a and b are constants.

Eqn. (8) will represent a parabola *only* if we show that the terms in Eqn. (8) corresponding to a and b of Eqn. (9) are indeed constant terms. We observe that θ_o is the launch angle. Hence both θ_o and $tan\theta_o$ have a constant value. Similarly v_{ox} stays constant for the entire duration of motion. And g of course *can never change*.

Eqn. (8), therefore, truly represents a parabolic trajectory. We have proved that the trajectories of *all types* of projectiles are parabolas!

11.9 The Time Dependent and the Time Independent Equations.

For solving most problems, we shall follow the format given above, where we use the. *time dependent* equation:

$$h = v_{oz}t + (1/2)\,g\,t^2 \qquad \text{.......... Eqn. (3)}$$

For a small number of problems, it will be to our advantage to use the time independent equation:

$$h = (tan\,\theta_o)\,x - \left\{\frac{(g/2)}{(v_{ox})^2}\right\}x^2 \qquad \text{......... Eqn (8)}$$

11.10 The Symmetrical Trajectory, or Trajectories with $h = 0$

In this section we shall develop some equations to determine and corelate various parameters for the benefit of a symmetrical trajectory. This preferential treatment is possible because for a symmetrical trajectory, at $x = R$, the z-mode displacement h is zero.

11.11 Range R

Setting $x = R$ and $h = 0$ in Eqn. (7) we get:

$$0 = \left(\frac{v_{oz}}{v_{ox}}\right) R - \left\{\frac{(g/2)}{(v_{ox})^2}\right\} R^2$$

Multiply both sides by v_{ox} and divide by R, to get:

$$0 = v_{oz} - \left\{\frac{(g/2)}{(v_{ox})}\right\} R$$

Solving for R, we get:

$$R = \frac{2 v_{ox} v_{oz}}{g} \qquad \text{..........(10)}$$

Eqn. (10) can directly be used to find R; but we can transform it into a different and equally useful equation. Knowing that $v_{ox} = v_o \cos\theta_o$ and that $v_{oz} = v_o \sin\theta_o$, we write for Eqn. (10):

$$R = \frac{2v_o^2(\sin\theta_o)(\cos\theta_o)}{g} = \frac{v_o^2(2\sin\theta_o\cos\theta_o)}{g}$$

From trigonometry:

$$2\sin\theta_o\cos\theta_o = \sin(2\theta_o)$$

we get, as *final* equation:

$$R = \frac{v_o^2 \sin(2\theta_o)}{g} \qquad \text{..........(11)}$$

11.12 Maximum Height h_{max}

This parameter is found by working in z-mode. Consider the *up* part of the motion. When at the top of its flight, the projectile is momentarily at rest. Hence $v_z = 0$. Now that one velocity is zero, we can use the short form equation, Eqn. (5). Replacing v_z by v_{oz}:

$$h = (v_{oz}^2)/(2g)$$

$$h_{max} = \frac{(v_{oz}^2)}{2g} = \frac{(v_o \sin\theta_o)^2}{2g} = \frac{v_o^2(\sin\theta_o)^2}{2g} = \frac{v_o^2(\sin^2\theta_o)}{2g}$$

$$h_{max} = \frac{(v_{oz}^2)}{2g} = \frac{v_o^2(\sin^2\theta_o)}{2g} \qquad \text{..........(12)}$$

11.13 Time of Flight t_{fl}

This parameter is also found by working in z-mode. Consider the *up* part of the motion again. At the top of the flight, $v_z = 0$. Using short form equation, Eqn. (2) and replacing v_z by v_{oz}:

$$v_{oz} = g t_{up}$$

We readily find that

$$t_{up} = \frac{v_{oz}}{g} = \frac{v_o \sin\theta_o}{g}$$

As this trajectory is symmetrical, $t_{up} = t_{down}$. The total time t_{fl}, is therefore, $2t_{up}$ or $2t_{down}$. We get:

$$t_{fl} = \frac{2v_{oz}}{g} = \frac{2(v_o \sin\theta_o)}{g} \qquad \text{..........(13)}$$

11.14 Time of Flight t_{fl} & Maximum Height h_{max}

To corelate t_{fl} **and** h_{max}, we need to eliminate v_{oz} between Eqns. (12) and (13). From Eqn. (13):

$$v_{oz} = \left(\frac{g}{2}\right)t_{fl}$$

and v_{oz}^2 is:

$$v_{oz}^2 = \left(\frac{g^2}{4}\right)t_{fl}^2$$

Insert this value of v_{oz}^2 in Eqn. (12), to get:

$$h_{max} = \frac{\left(\frac{g^2}{4}\right)(t_{fl}^2)}{2g} = \left(\frac{g^2}{4} \times \frac{1}{2g}\right)(t_{fl}^2) = \left(\frac{g}{8}\right)(t_{fl}^2) = \left(\frac{9.80}{8}\right)(t_{fl}^2)$$

Or:

$$h_{max} = 1.225(t_{fl}^2) \qquad \text{..........(14)}$$

11.15 Maximum Height h_{max} & Range R

To corelate h_{max} and R**,** we recall Eqn. (10) and then carry out some simple algebraic manipulations:

$$R = \frac{2v_{ox}v_{oz}}{g} \qquad \text{......... Eqn. (10)}$$

Dividing both sides by v_{ox}.

$$\left(\frac{R}{v_{ox}}\right) = \frac{2v_{oz}}{g}$$

Multiplying both sides by v_{oz}:

$$\left(\frac{R}{v_{ox}}\right)(v_{oz}) = \frac{2(v_{oz}^2}{g} \qquad \text{rearranging:} \qquad R\left(\frac{v_{oz}}{v_{ox}}\right) = \left(\frac{2}{g}\right)(v_{oz}^2)$$

From Eqn. (12),

$$v_{oz}^2 = 2g(h_{max})$$

Plug in this value of v_{oz}^2 in Eqn. (15) to get:

$$R\left(\frac{v_{oz}}{v_{ox}}\right) = \left(\frac{2}{g}\right)2g(h_{max}) = 4(h_{max})$$

As

$$v_{oz}/v_{ox} = \tan\theta_o$$

we get as our final equation;

$$R(\tan\theta_o) = 4(h_{max}) \qquad \text{..........(15)}$$

11.16 Range R & Time of Flight t_{fl}

To corelate R and t_{fl}**,** simply plug in the value of h_{max} from Eqn. (14) into Eqn. (16)**:**

$$R(\tan\theta_o) = 4(1.225)(t_{fl}^2)$$

Or,

$$R(\tan\theta_o) = 4.90(t_{fl}^2) \qquad \text{..........(16)}$$

11.17 Summary; (a) Formulae For All Types of Trajectories

(a) Time Dependent (x-mode): $v_{ox} = v_x = d/t$

(b) Time Dependent (z-mode): $h = v_{oz}t + (1/2)\,g\,t^2 \qquad h = (1/2)\,g\,t^2$

(c) Time Independent $h = (\tan\theta_o)\,x - \left\{\dfrac{(g/2)}{(v_{ox})^2}\right\}x^2$

Summary (b) Formulae Specific to Symmetrical Trajectories

(d) Time of Flight (general) $t_{fl} = \dfrac{2v_{oz}}{g} = \dfrac{2(v_o \sin\theta_o)}{g}$

(e) Maximum Height $h_{max} = \dfrac{(v_{oz}^2)}{2g} = \dfrac{v_o^2(\sin^2\theta_o)}{2g}$

(f) Range $R = \dfrac{v_o^2 \sin(2\theta_o)}{g}$

(g) Time of Flight & Maximum Height $h_{max} = 1.225(t_{fl}^2)$

(h) Maximum Height & Range $R(\tan\theta_o) = 4(h_{max})$

(i) Range & Time of Flight $R(\tan\theta_o) = 4.90(t_{fl}^2)$

11.18 Examples

We now present a set of problems and their solutions. Symmetrical trajectories are taken up first. Time independent problems will be found at the end.

Example 1

A football leaves the ground at angle of 32° above the horizontal at 18 m/s. Find (i) the range, (ii) maximum height to which it rises, (iii) the time for which it is in air, and (iv) the impact velocity. Let *air* be an unneighbor.

Solution

This is a symmetrical trajectory so we can bypass the format and use special equations, listed in **Section 11.17.**

(i) Range: use Eqn. (11):

$$R = \frac{v_o^2 \sin(2\theta_o)}{g} = \frac{18^2 \sin(64)}{9.8} = \mathbf{29.715}\ m$$

(ii) Maximum height: Use Eqn. (12);

$$R(\tan\theta_o) = 4(h_{max}) \qquad (29.715)(\tan 32) = 4(h_{max}) \qquad h_{max} = \mathbf{4.642}\ \text{m}$$

(iii) Time of flight; use Eqn. (13)

$$h_{max} = 1.225(t_{fl}^2) \qquad 4.642 = (1.225)(t_{fl}^2) \qquad t_{fl} = \mathbf{1.947}\ \text{sec}$$

(iv) Impact velocity. Since trajectory is symmetrical and air resistance is ignored, impact speed. will be the same as the launch speed. Angle of impact will have the same magnitude as the angle of launch. However, it will be in the 4th quadrant.

18 m/s, at 32° in 4th quadrant

Example 2

A small plane is flying horizontally at 225 km/h over a lake, at a height of 90 m above water surface. With what initial speed (vertically up or vertically down) did a passenger in the plane throw a small iron ball that successfully hit an old disused raft floating on the surface of water? Assume that the plane's horizontal distance from the raft was 250 m at the time the ball was thrown.

Solution:

Step (1) Dump data in the two bins

Table 2: The Bins & The Dumping

x-mode	**z-mode**
$v_{ox} = v_x = d/t$	$h = v_{oz}t + (1/2)gt^2$
$d = 250$ m $v_{ox} = \frac{225}{3.6} = 62.5$ m/s	$v_{oz} =$ $h = -90$ m

Step (2) Find t_{fl}**:**

This we can do in x-mode only:

$$v_{ox} = d/t \qquad 62.5 = 250/t_{fl} \qquad t_{fl} = 4.000 \text{ sec}$$

Step (3) Carry t_{fl} into the z-bin

$$h = v_{oz}t + (1/2)gt^2 \qquad -90 = (v_{oz})(4.000) + (1/2)(-9.8)(4.000^2)$$

Solving for v_{oz}, we get:

$$v_{oz} = \mathbf{-2.900} \text{ m/s}$$

The passenger threw the ball vertically downward with a speed of 2.900 m/s

Example 3

At what angle (with respect to the horizontal) should a projectile be launched such that the maximum height to which it rises h_{max}, equals its horizontal range R? The projectile makes a symmetrical trajectory and air is an unneighbor.

Solution:

Recall: $R(\tan\theta_o) = 4(h_{max})$ As h_{max} equals R, the two cancel out giving us:

$$\tan\theta_o = 4 \qquad \boldsymbol{\theta_o = 75.964°}$$

Example 4

Water is streaming out from a hose poised 1.80 m above ground level, at an angle $\theta_0 = 27°$ with the horizontal. It is suddenly switched off. It is noticed that the last of the water that left the hose, took 1.463 sec (measured electronically) to hit the ground below. With what initial speed v_o did the water leave the hose?

Solution:

As t_{fl} is already given, the 3-step format is not applicable. We use the time dependent equation in the z-mode and find v_{oz} directly.

$$-1.80 = (v_{oz})(1.463) + (1/2)(-9.80)(1.463^2) \qquad v_{oz} = 5.9384 \text{ m/s}$$

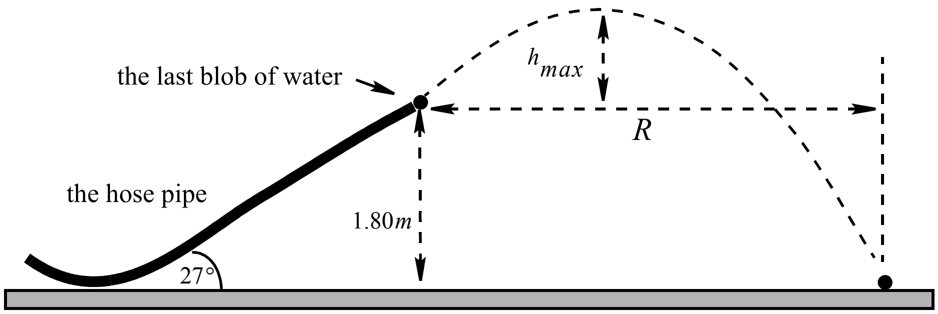

Fig (7) Diagram for Problem # 4

Now: $v_{oz} = (v_o)(sin\theta_o)$ $\quad 5.9384 = (v_o)(sin\ 27°)$ $\quad$ $\boldsymbol{v_o = 13.080}$ **m/s**

The problem, as stated, is solved. We can, additionally, find the range R, the horizontal distance travelled by the water blob from the end of the nozzle.

$$R = (v_{ox})(t_{fl}) \qquad R = (v_o\, cos\theta_o)(t_{fl})$$

$$R = 13.080 \times cos 27° \times 1.463 \qquad \boldsymbol{R = 17.0508\ m}$$

As for the maximum height,

$$h_{max} = \frac{(v_{oz}^2)}{2g} = \frac{5.9384^2}{19.60} = 1.799\ \text{m}$$

and total height of the water blob above ground level is

$$h_{total} = 1.799 + 1.8 = \mathbf{3.599\ m}$$

Example 5

A football leaves the hands of the quarterback at a height of 0.050 m above level ground with a speed $v_o = 20$ m/s, at $\angle\theta_o = 25°$ with respect to the level ground. It is caught in mid-air by a receiver at a height of 1.50 m above ground. (i) How far away (horizontally) was the receiver from the initial position of the quarterback? (ii) with what velocity was the ball moving at the time it was caught? (iii) Find the total time for which the ball was in air.

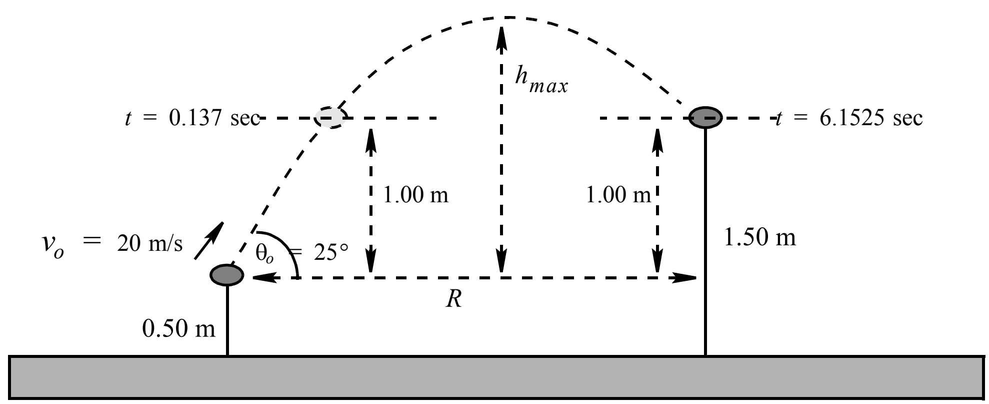

Fig (8) Diagram for Problem # 5

Step (1): The Bins

Table 3: The Filled Bins,

x-mode	**z-mode**
$v_{ox} = v_x = d/t$	$h = v_{oz}t + (1/2)gt^2$
$v_{ox} = v_x = 20cos25° = 18.126$ m/s	$v_{oz} = 20sin25° = 8.4524$ m/s $h = +1.00$ m $g = -9.8 \ (m/s^2)$

Step (2): Find t_{fl}

Using the time-dependent equation for z-mode (the column for z-mode)

$$h = v_{oz}t + (1/2)gt^2 \qquad 1 = (8.4524)(t) + (1/2)(-9.8)(t^2)$$

This gives us a quadratic equation in t:

$$4.9t^2 - 8.4524t + 1 = 0 \qquad t = \frac{8.4524 \pm \sqrt{8.4524^2 - 4 \times 4.9 \times 1}}{2 \times 4.9}$$

$$t = (8.4524 \pm 7.2002)/(9.80) = 6.1525 \text{ sec} \quad \text{or} \quad 0.137 \text{ sec}$$

We shall accept the value: $\boldsymbol{t = 6.1525}$ **sec** $\boldsymbol{= t_{fl}}$ answer for part (iii)

Step (3): Find other answers

To find R, as shown in the diagram, use the equation in the column for x-mode:

$v_{ox} = v_x = d/t \qquad 18.126 = R/6.1525 \qquad$ $\boldsymbol{R = 111.520}$ **m** ..... part (i)

To find the answer for part (ii), we need v_x and v_z. Now $v_x = v_{ox} = 18.126$ m/s. The value of v_z comes from the determinant and is

$$\sqrt{8.4524^2 - 4 \times 4.9 \times 1} = 7.2002 \text{ m/s}$$

The impact velocity $v_{impact} = v$ is:

$\sqrt{(v_x^2 + v_x^2)} = \sqrt{(18.126^2 + 7.2002^2)} =$ **19.504 m/s** .... part (ii)

The angle $\angle\theta$ that the impact velocity makes with the x-axis is given by

$\theta = tan^{-1}\left(\frac{v_z}{v_x}\right) = tan^{-1}\left(\frac{7.2002}{18.126}\right) =$ **21.664° 4th quadrant** .... part (ii)

Additional Remarks

We did not find things like maximum height, as they were not required. But they could be found easily. However, it should be pointed out that we found two values of time. This is because the ball reaches the height $h = 1$ m, twice, as shown in Fig (8). Should a player of the opposite team be there and succeeds in catching the ball, an interception would occur.