Chapter 10

The Activity Region

Activators as Vectors

10.0   Introduction

The agencies that act on objects in activity regions are called *activators* because they activate the processes that cause objects to change their equilibrium states. In previous chapters, we introduced two agencies, (i) velocity, and (ii) acceleration. We further found that these two agencies were in fact *vectors*. We now state that all activators must indeed be vectors.

Only vectors can serve as activators in activity regions.

Non-vectors (*scalers*) need not apply for *activator* jobs. They are inherently ineligible.

10.1   Defining Vectors

Vectors, in physics, are entities that require an appendage of direction (orientation in space) for their meaningful description and total understanding. Forces are good examples. We always apply a force in a particular direction; a hammer must be moved in the direction of the nail. A car travelling toward a specific destination possesses the vector property known as velocity. Momentum, acceleration, and displacement are some other examples of vectors. It is appropriate to tell you at this time, that *time* that always flows in the forward direction and an electric current that flows along the wire, do not qualify to be called vectors. The number of vectors in physics is indeed limited but the ones that are there, need special considerations and ignoring their special needs is sure to cause chaos.

With vectors, you may be up for some surprises. We may, for example, tell you that in vector-land $2 + 2 = 2$ has distinct possibility of being true; or that if 71% of an applied force pulls a crate along a flat floor, then at the same time, 71% of the same force doesn't pull the crate; the floor remaining an un-neighbor all the time.

Before we get totally bewildered, let's get familiar with their characteristics.

10.2   Role of Vectors

Vectors in physics, always act on objects/systems of objects. Any number of vectors (of the same genre) may act on a single object (or system of objects) simultaneously.

10.3   Representing Vectors

Vectors, for reasons of the appendage of direction, cannot be represented by simple numbers. Numbers only tell the magnitudes and give us no clue as to how this number is oriented in space. An intuitive way of accomplishing the representation of both, the magnitude and the direction, is to use an *arrow.* Arrows have a *tip* and a *toe.* The length of the arrow, from tip to toe, will be representative of the magnitude of the vector and the arrowhead (the *tip*) will be indicative of the orientation of the vector in space. Fig (1) below shows eight vectors represented by arrows.

When representing vector by a letter in the text, we may (i) make the letter bold, or (ii) underline it, or (iii) draw a small arrow over the letter. In this text we shall express vectors

by bold letters. Now, what if you were to write a vector in your notebook? Realizing that it will be too much to ask you to make every vector a bold letter, the physics community has decided to ask you to simply underline it. Books will write vectors as: **a**, **P**, **m**, **D**, **y**, etc. You and I will write them as a̲, P̲, m̲, D̲, y̲, etc.
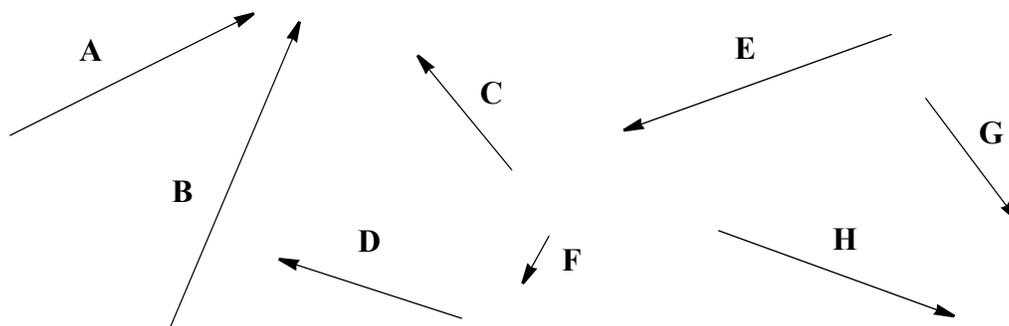

Fig (1). Vectors: Geometrical Representation

Arrows (and hence the vectors) are directed from the toe to the tip. The magnitude and the direction of a vector (and hence that of an arrow) are both sacred entities and cannot be violated. We, therefore, cannot rotate an arrow or pull or push its tip or toe up or down in order to make its length larger or smaller. We can, of course move an arrow parallel to itself and put it wherever we wish. We may for example put it in a reference frame. Fig (2) shows the eight vectors of Fig (1), relocated so as to lie at the origin of an (x,y) reference frame. Each has been carefully moved parallel to itself so as to conserve its identity.
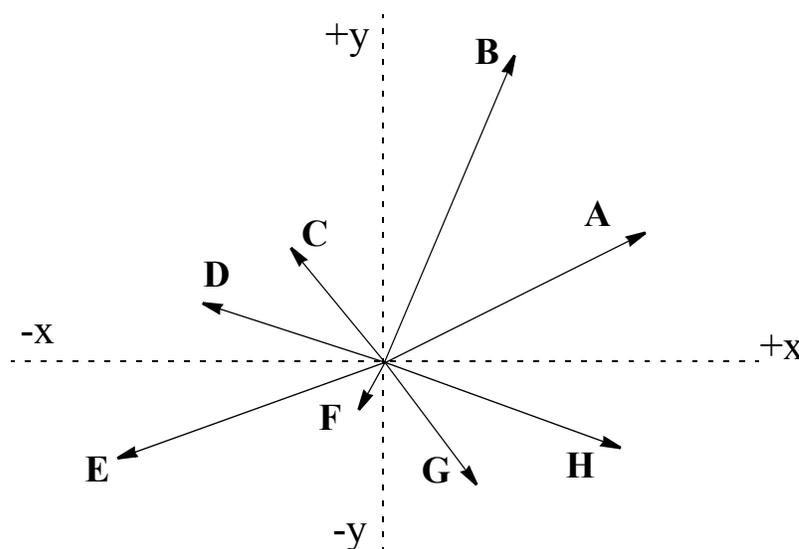

Fig (2) Vectors of Fig (1) Have Been Repositioned.

10.4   Vectors Act on Our Objects

Vectors in physics always act on an object. The center of mass of the object is placed at the origin of a reference frame (such as shown in Fig 2 above). According to this diagram, eight vectors are acting on the object. Will the object be affected by all eight vectors, in all eight directions, all at the same time? The answer is *no*. This is because a rigid body cannot get affected in several directions, all at once, without breaking up into pieces. For the object to get affected by the eight vectors, the eight vectors must consult with one another and form one *overall* vector. In other words, they should hire a *representative* vector that

would truly represent all of them. This representative vector will be an *additional* vector, or an *outside* vector or the *ninth* vector.

The representative vector that truly represents a given set of vectors, is called the *resultant* vector. The process of finding the *resultant* vector is called the process of *combination*. When a number of vectors act on an object, we first combine them to find their resultant vector and then study the effect of this resultant vector (a single vector) on the object. Once the resultant vector has been found, *all the original vectors become redundant* and we must ignore them as if they never existed. *Push them off the cliff*, so to say.

10.5   How Do We *Combine* Vector?

Vectors may be combined either geometrically or analytically. Use of geometry necessarily makes us dependent on rulers and protractors. Thus it may not be a highly desirable approach. However, this is how we shall begin. Once we have understood the *procedure*, we will convert geometry into algebra and trigonometry to free ourselves from rulers and protractors.

10.6   The *Geometrical* Method

The geometrical procedure of combining a set of given vectors, consists of arranging them successively (one after the other), by picking them in *any* order. To arrange successively, we shall pick one vector from the pack and, moving it parallel to itself, reposition it in a reference frame. The toe of the vector will be at the origin. At the tip of this vector, we shall bring and place another reference frame. The toe of the second vector (that we choose to pick from the pack) will lie at the origin of this reference frame. At the tip of the second vector, we shall institute a third reference frame and the toe of the third vector will be placed at the origin of this reference frame. Each vector must be moved parallel to itself so that its magnitude and direction don't get damaged.
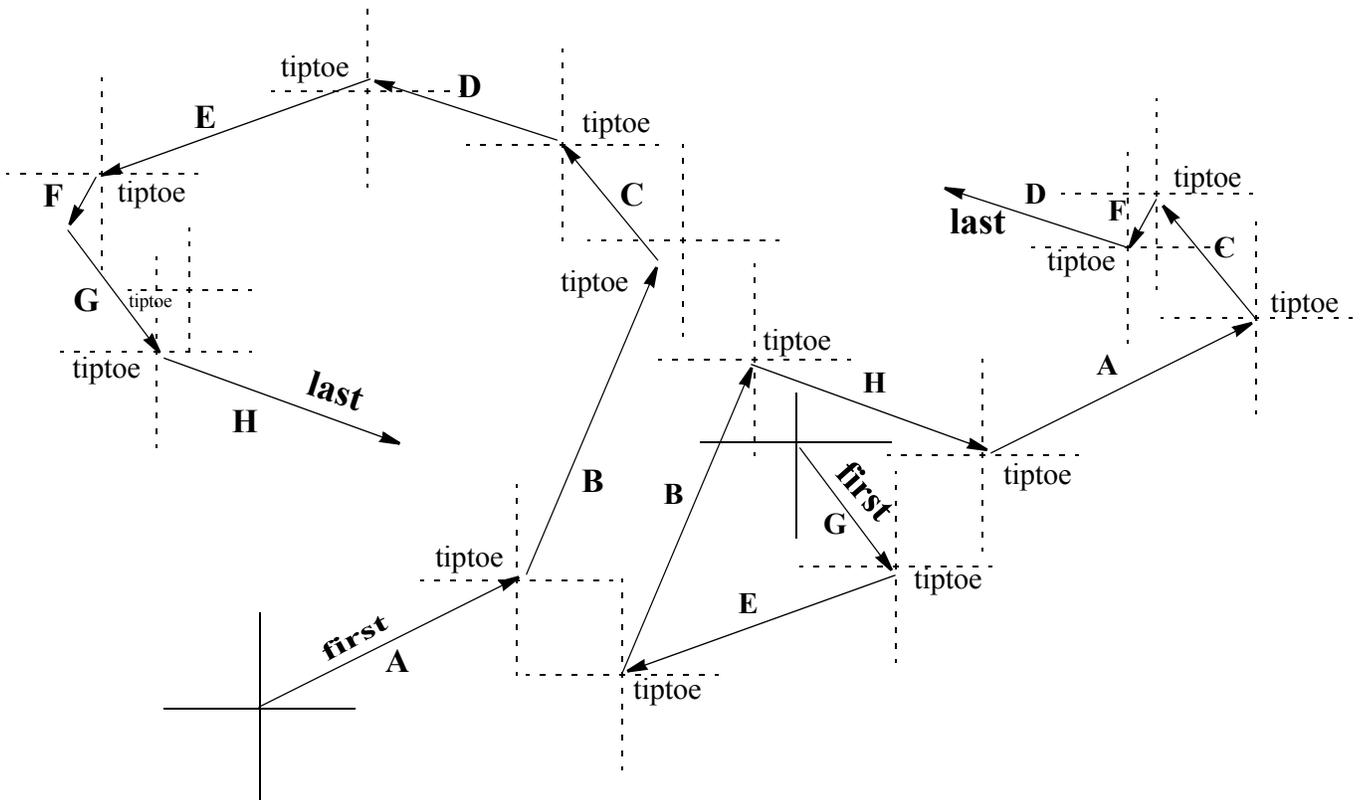

Fig (3) Combining or Tip-toeing Vectors By Picking Them in Different Order

In the diagram above (Fig 3), the eight vectors of Fig (1) have been combined by picking vectors in two different modes. The resultant vector and its orientation is found to be identical in the two cases. You would have noticed that every time two vectors are connected together, one after the other, a *tip* and a *toe* meet. We may say that we *tip-toe* the vectors. Or, that in order to combine vectors we go *tiptoe, tiptoe.....*

When all vectors have been so placed, then we connect the toe of the first vector with the tip of the last vector. The arrow so formed is our resultant vector. The toe of this arrow meets the toe of the first vector and its tip meets the tip of the last vector. You should notice that two *toes* are glued to the origin of the first reference frame. The orientation of the resultant vector can be measured in terms of the angle that the resultant vector makes with the x-axis, in the very first reference frame. A resultant vector is thus identified as the odd vector which doesn't tiptoe with other vectors. Instead it is a *tip-tip, toe-toe* vector. Sometime we think that we are smart and we do not have to bring a new reference frame every time. Using fresh reference frames every time is very important and is strongly recommended.

The resultant vectors determined geometrically in the two cases, are equal to one another in all respects. Thus the order of picking vectors for combining them, is totally immaterial. This is, in fact, a basic property of vectors. We show it below for three vectors: *A* , *B* and *C* . There are six different permutations for combining them. The property, known as the **Commutative Law**, holds for any number of vectors.

$$A + B + C \ = \ B + C + A \ = \ C + A + B \ = \ A + C + B \ = \ B + A + C \ = \ C + B + A \ = \ R$$

If two or more vectors do not obey the commutative law, then they are not vectors!
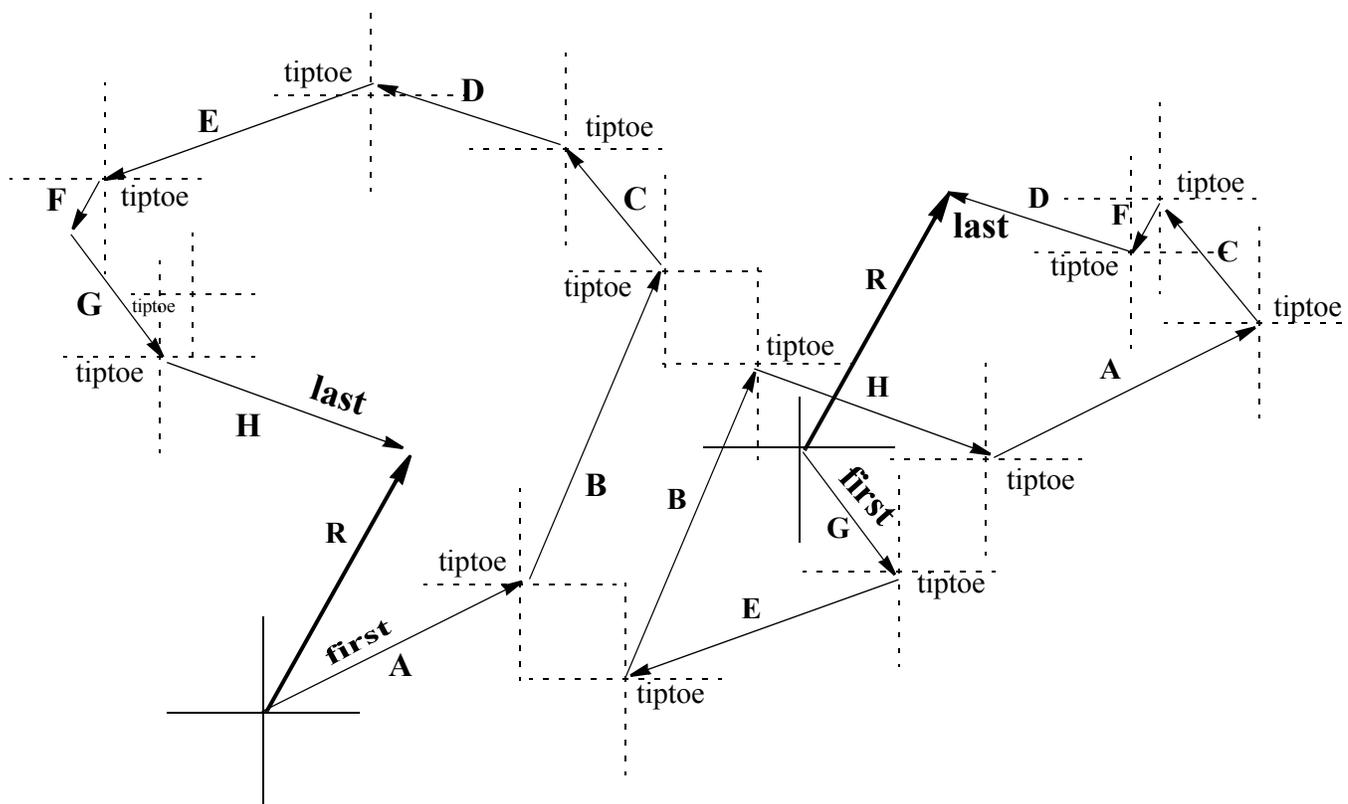

Fig (4) Combining Vectors to Form a Resultant Vector

10.7 Combining Two Vectors

(a) Combining Two Parallel Vectors, i.e. Vectors that have the Same Direction).

In this case, upon tiptoeing, we find that we simply *added* the magnitudes of vectors *scalerly*. As the vectors had the same direction, the direction before and after combining, did not change. Some illustrations are given below. We may formulate the following rule:

Rule #1

If two or more vectors having the same direction are to be combined, then their magnitudes will be added scalerly. The direction of the resultant vector will be the same as that of any of the individual vectors.
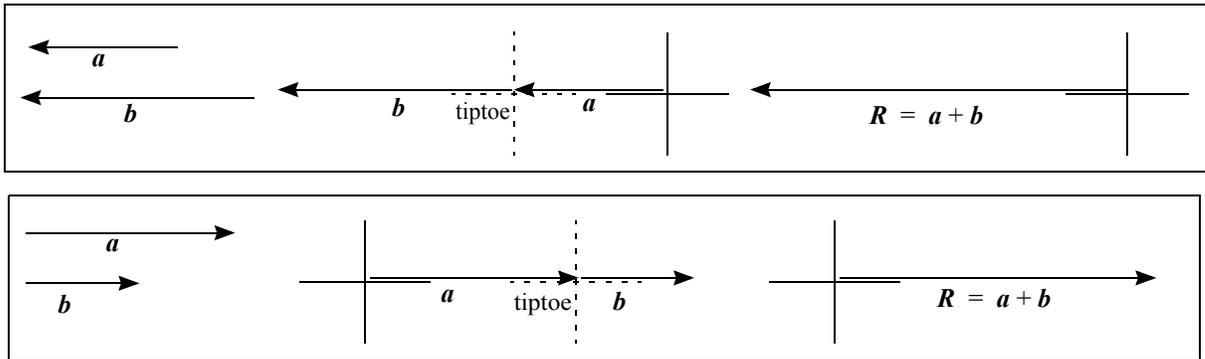

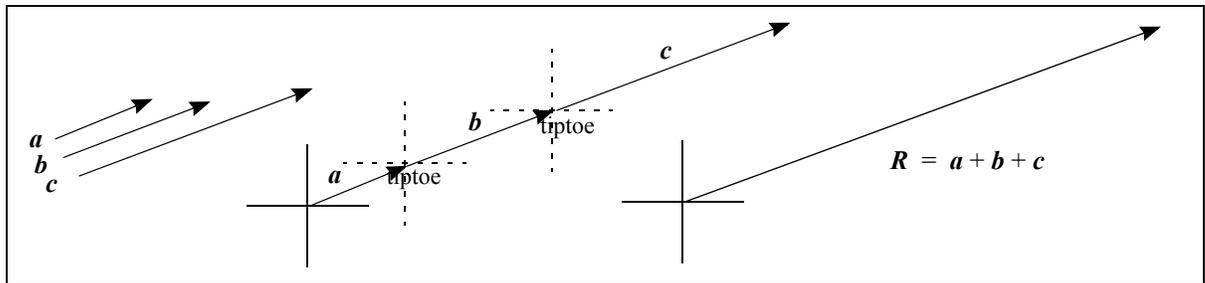*Fig (5a & b) Combining Vectors That Have The Same Direction*

Fig (5c) Combining Vectors That Have The Same Direction

(b) Combining Two Anti-Parallel Vectors, i.e. Vectors that have Opposite Directions).

In this case, upon tiptoeing, we find that we simply subtracted the magnitudes of the vectors (smaller from the larger) scalerly. It will also be found that the direction of the vector with larger magnitude prevailed. Also note that even though we apparently subtracted one vector from another, we still write $a + b$. The positive sign represents *combination* of vectors. *Remember, we always combine vectors.*
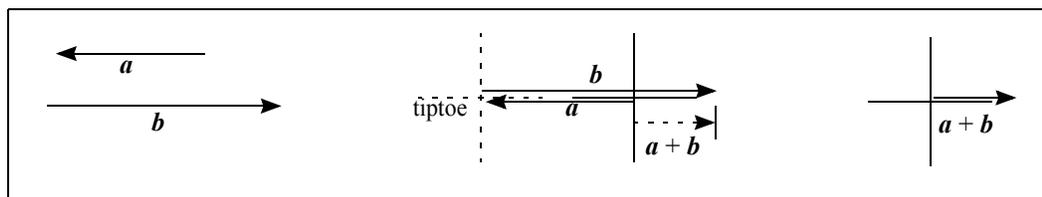

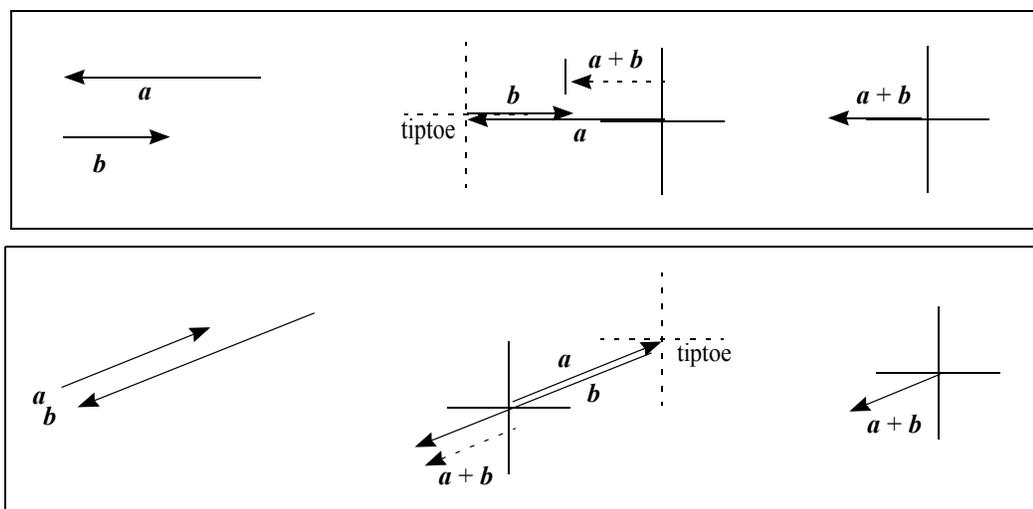*Fig (6a) Combining Vectors That Have Opposite Directions*

Fig (6 b, c) Combining Vectors That Have Opposite Directions

We now formulate our second rule:

Rule #2

If two vectors have opposite directions, their magnitudes may be subtracted scalerly. The direction of the resultant vector will be that of the vector with larger magnitude.

These rules apply to any number of vectors. Figs (5) and (6) show the above described combinations for three sets of vectors in each case. The reference frames have been displaced slightly for vectors to be visible.

(c) Combining "Just-Any-Two" Vectors

In this case, upon tiptoeing, we find that the two vectors and their resultant form a triangle. The triangle can be an acute angled triangle, an obtuse angled triangle or a right angled triangle. As we can always apply sine and cosine laws to triangles, we find that combining just-any-two vectors is a very practical proposition.

(a) two "given" vectors   (b) the two vectors are tiptoed   (c) the "resultant" vector is found
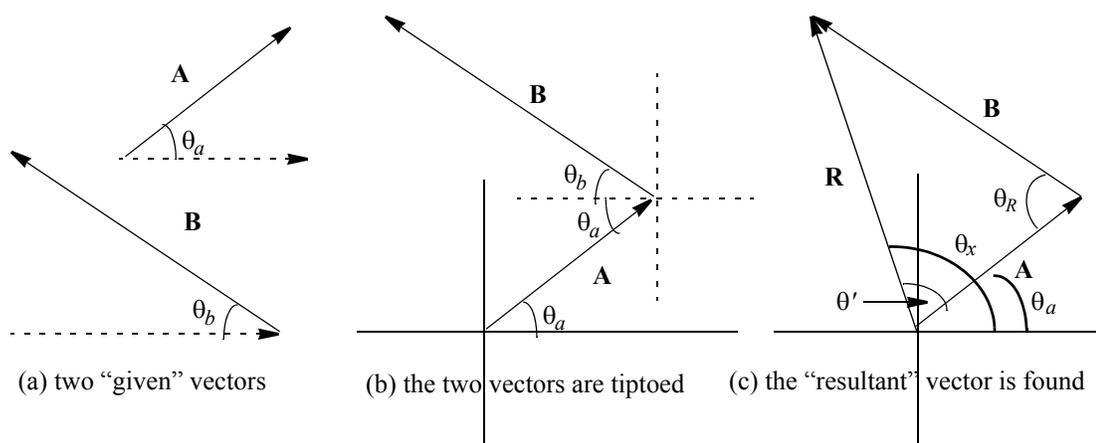
Fig (7) Using Cosine & Sine laws for Combining Just-Any -wo Two Vectors

As vectors necessarily come with their directions, it is always possible to determine the angle opposite the resultant vector. Fig (7) shows a typical combination of two vectors and illustrates the use of the sine and cosine laws for finding the resultant vector (magnitude and direction).

The resultant vector **R**, is determined by using the cosine law:

$$R^2 = A^2 + B^2 - 2AB\cos\theta_R \qquad\qquad ....(1)$$

where $\theta_R = \theta_a + \theta_b$. We find θ' by using the sine law:

$$\frac{sin\,\theta'}{B} = \frac{sin\,\theta_R}{R}$$

The angle of the resultant vector R with respect to the x-axis, θ_x, is (as shown) given by:

$$\theta_x = \theta' + \theta_a \qquad\qquad ....(2)$$

(d) Special Case: The Two Vectors Are At Right Angle to Each Other.

If the two vectors of the preceding section, are at right angles to each other, the angle θ_R, opposite the third side which happens to be the resultant vector, will be 90°. Setting $\theta_R = 90°$ in Eqn (1) we find that the cosine law reduces to Pythagorean theorem.
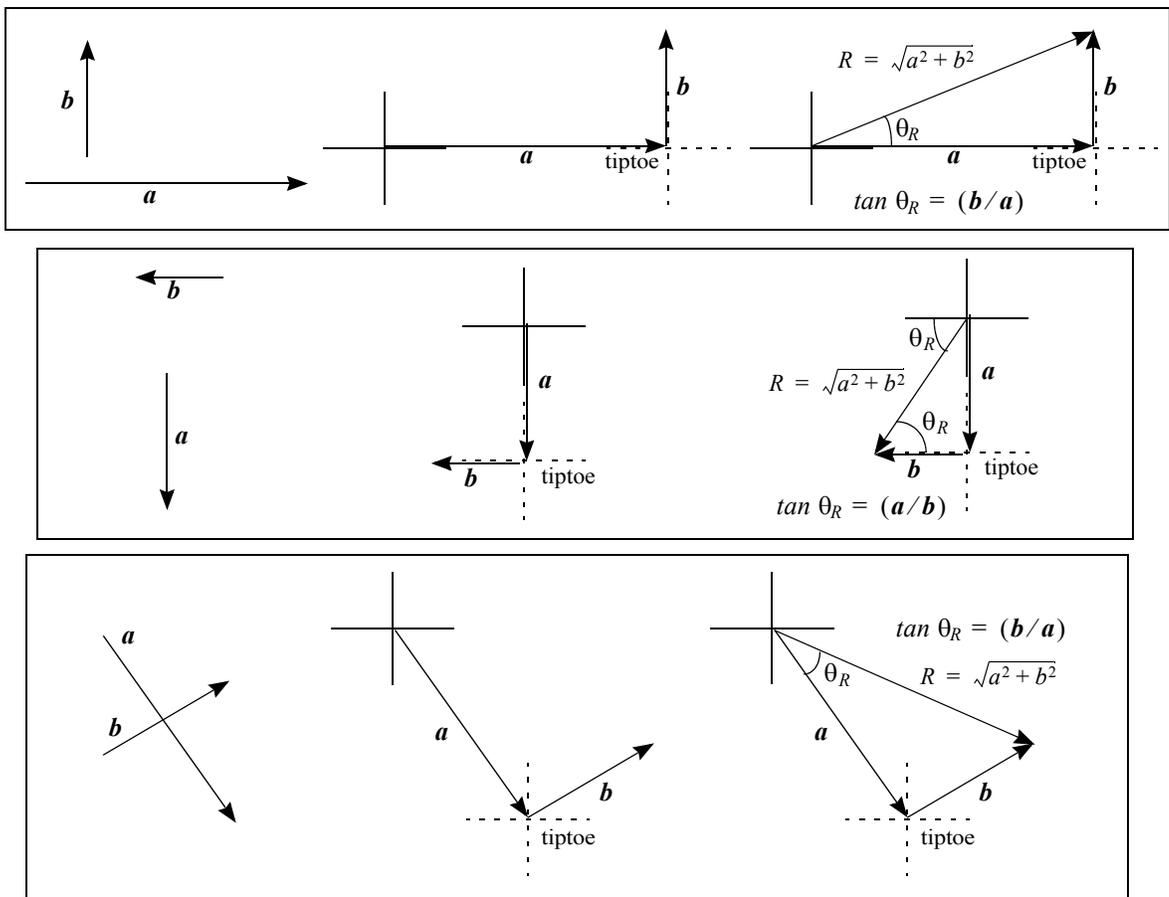
Fig (8) Combining Vectors That Are Perpendicular to One Another

We shall now write down Rule #3 for combining vectors:

Rule #3

If two vectors are perpendicular to each other then they can be combined using the Pythagorean theorem. The Resultant vector is the hypotenuse of the triangle and its direction is found from the ratio of the two vectors:

$$R = \sqrt{a^2 + b^2} \qquad tan\,\theta = \left(\frac{y}{x}\right) \qquad\qquad ....(3)$$

10.8   Inquisitivity: What If?

The ease with which parallel, anti-parallel and perpendicular vectors can be combined, makes us think *what if* all vectors were parallel, anti-parallel and/or perpendicular to one another, then it would be *so very* easy to find their resultant! We will just use scaler addition, subtraction and the Pythagorean theorem. There will be no protractors, no rulers and no reference frames and no *tiptoes*!

10.9   Intuitivity: How About?

The *what if* became reality with the intuitive idea that we treat every vector as the hypotenuse (of a right angled triangle, naturally) and let it grow two sides that would be at right angles to each other. A hypotenuse can, however, generate sides in many, many different ways Following diagram shows some possibilities.
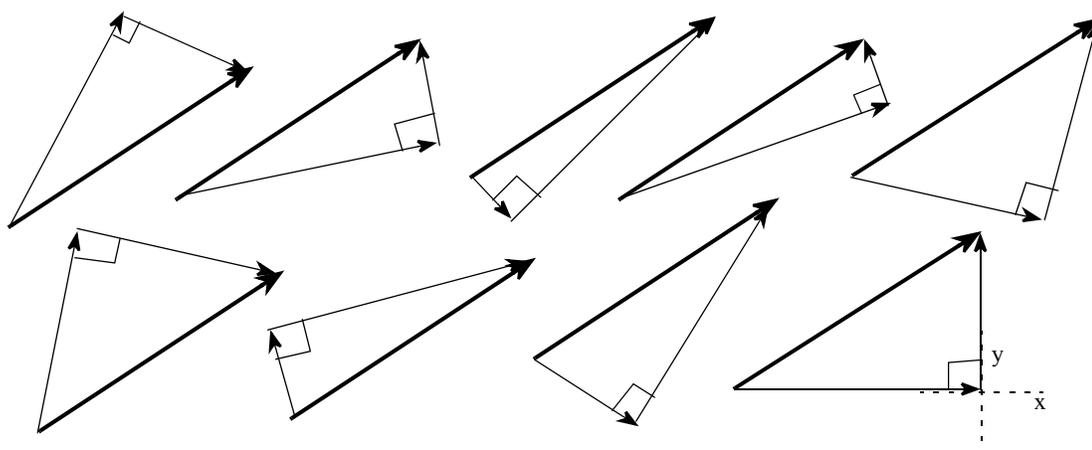
Fig (9) A Hypotenuse Grows Two Sides

As random generation of sides by the concerned vectors will lead to anarchy, we impose a constraint: the hypotenuse must grow one of its two sides parallel to our x-axis! This necessarily puts the other side along the y-axis of our reference frame. If all hypotenuses grew sides such that one side was always parallel (or anti-parallel) to the x-axis, then we would simply use Rule #s 1 & 2 and add (or subtract) them scalerly. The result will be just *one* vector which could be called the *Resultant x-vector*. The same will apply to the second set of sides of the hypotenuses. These will all be parallel or anti-parallel along the y-axis. We would then use Rule #s 1 & 2 again, and add or subtract them scalerly. The result will again be just *one* vector which could be called the *Resultant y-vector*. Most wonderfully now, the two resultant vectors will be found to be at right angles to one another, making it possible for us to use Rule #3 and get the resultant vector of all given vectors! Applying this idea to the 8 vectors of Fig (1), we get:
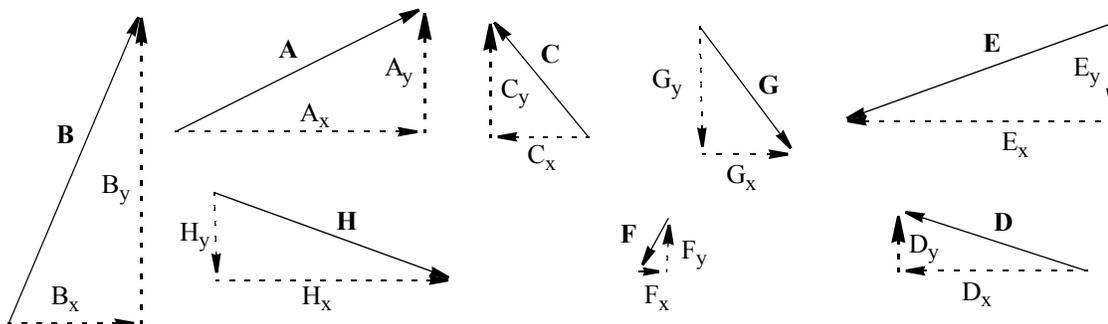
Fig (10) Vectors as "Hypotenuses"; Growing "Sides"

We next apply Rule #s 1, 2 and 3 to get the resultant vector (Fig 11). As can be seen, this resultant matches exactly with the ones found earlier and shown in Fig (4).
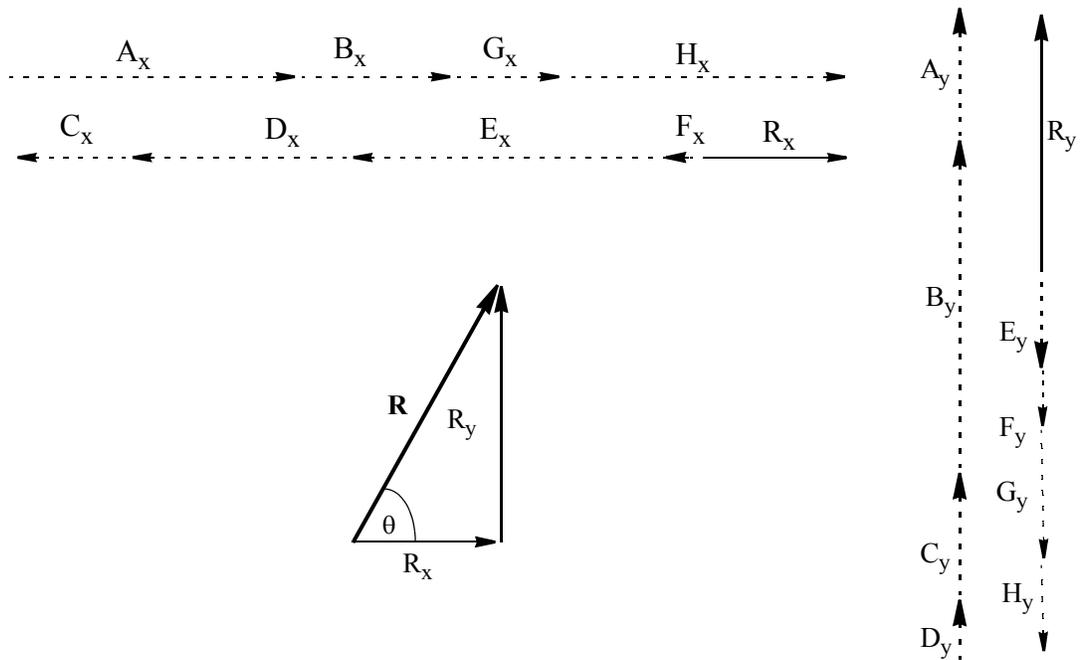

Fig (11) Combining "Sides" Scalerly & Applying Pythagorean Theorem To Get "Resultant Vector"

10.10   The Analytical Procedure

The analytical method can now be evolved with the help of elementary trigonometry: For any right angled triangle if the hypotenuse and the angle that the hypotenuse makes with the x-axis, $\angle\theta_x$, be given, then the two sides of the right angled triangle are given by

$$adj = (hyp)(cos\theta) \quad \text{and} \quad opp = (hyp)(sin\theta)$$

where *hyp* is hypotenuse, *adj* is the adjacent side, parallel to the x-axis, and *opp* is the opposite side which is parallel to the y-axis.

If the hypotenuse represents vector *A* (say), then the adjacent side that this hypotenuse grows parallel to our x-axis, may be called A_x. The opposite side may correspondingly be called A_y. We then have the following two equations:

$$A_x = A\cos\theta \qquad\qquad ....(4)$$

$$A_y = A\sin\theta \qquad\qquad ....(5)$$

A_x and A_y are called the *components* of vector *A* . It should be clearly understood that the components A_x and A_y are vectors too! While combining a number of x- or y-components, we may (temporarily) ignore their vector nature because they are being added or subtracted scalerly using Rules #s 1 & 2.

*This, boys and girls, is the "end all and be all" of the analytical technique.*

A knowledge of the vectors (magnitudes: A, B,.... and directions: θ_a, θ_b,....) enables us to determine the resultant vector (magnitude R and direction θ_R) using mathematics. No geometrical drawings are needed; (although a rough sketch is always helpful).

10.11   Specifying Directions of Vectors In Problems

   (i) With respect to the positive x-axis, in the range 0° to 360°.

As an example consider the following 5 vectors:

F_1 at 33°, F_2 at 140°, F_3 at 215°, F_4 at 300°, and F_5 at 345°
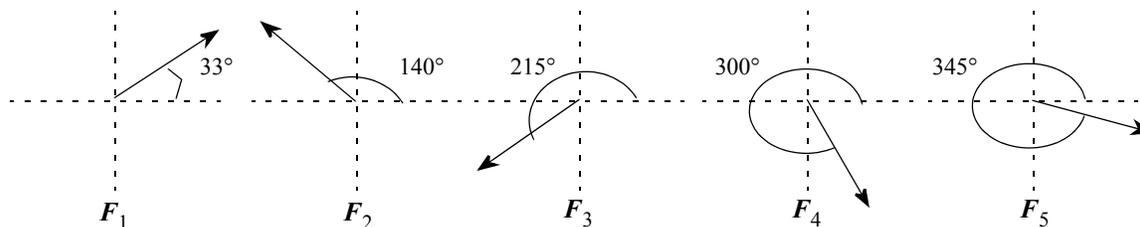
Lengths of the arrows do not reflect their magnitudes; which, at this point, is unspecified.

Fig (12) Directions of Vectors Specified with Respect to the Positive x-axis

In this case, we shall find R_x and R_y as:

$$R_x = F_1 \cos 33 + F_2 \cos 140 + F_3 \cos 215 + F_4 \cos 300 + F_5 \cos 345$$

$$R_y = F_1 \sin 33 + F_2 \sin 140 + F_3 \sin 215 + F_4 \sin 300 + F_5 \sin 345$$

You will notice that we *added* all components without any consideration of their being parallel or anti-parallel. This is because the calculator finds the correct algebraic sign for all vector components and adds and subtracts accordingly. For example, $\cos 140°$ equals -0.7660. This shows that vector component $F_2 \cos 140$ is anti parallel and as such, it will automatically get subtracted. Same is true for y-components. Consider $\sin 300$. It equals -0.8660. The vector component $F_4 \sin 300$ is, therefore, anti parallel and as such will be subtracted.

(ii) With respect to the quadrant in which the vector lies

The vectors of the above example will now be specified as

F_1 at 33°, 1st quadrant, F_2 at 40°, second quadrant, F_3 at 35°, 3rd quadrant

F_4 at 60°, 4th quadrant, F_5 at 15°, 4th quadrant.
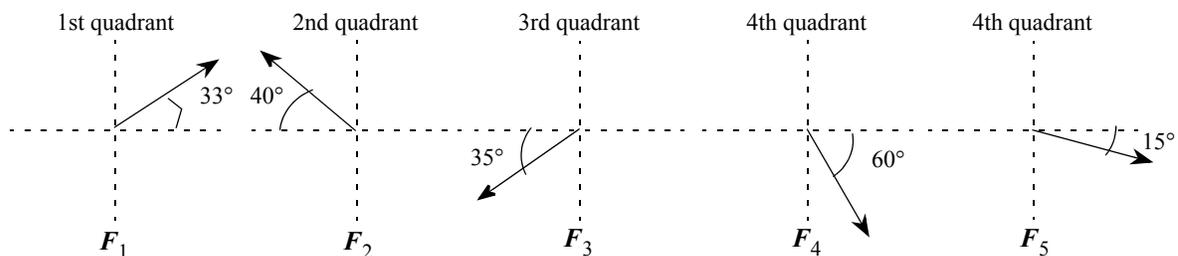
Lengths of the arrows do not reflect their magnitudes; which, at this point, is unspecified.

Fig (13) Directions of Vectors Specified with Respect to their Respective Quadrants

In this case, it is our duty to assign positive and negative signs to the components. It should be pointed out that x-component of vectors in the second and third quadrants lie along negative x-axis and hence carry a negative sign. Similarly y-component of vectors in the third and fourth quadrants lie along negative y-axis and hence carry a negative sign. Following diagram gives the signs of vector components in different quadrants.
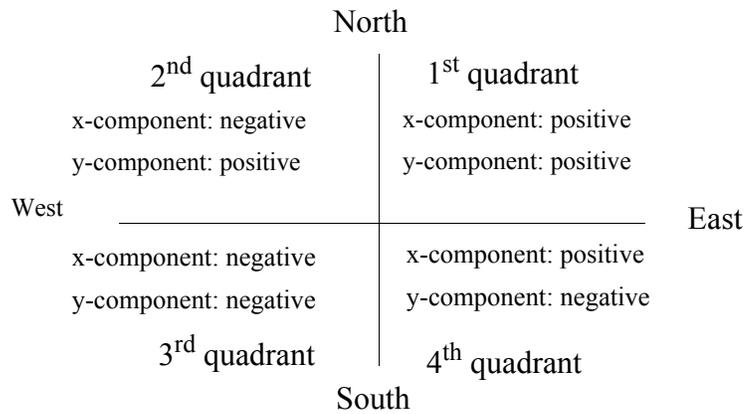
North

2nd quadrant

x-component: negative

y-component: positive

1st quadrant

x-component: positive

y-component: positive

West

East

x-component: negative

y-component: negative

x-component: positive

y-component: negative

3rd quadrant

4th quadrant

South

Fig (14) Algebraic Signs of Components of Vectors

(iii) With respect to geographical directions.

As shown in Chapter (xix) and Fig (13) above, angle may be given as "north of east" or "south of west etc. The vectors of the above example, may be given as:

F_1 at 33°, north of east, F_2 at 40°, north of west,

F_3 at 35°, south of west F_4 at 60°, south of east,

F_5 at 15°, south of east.
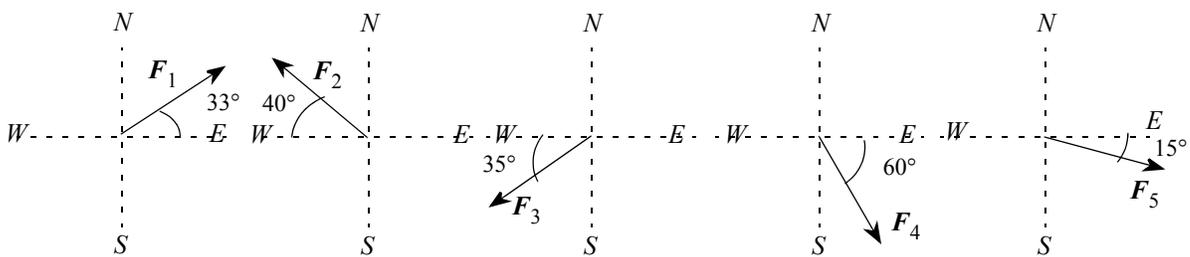
Lengths of the arrows do not reflect their magnitudes; which, at this point, is unspecified.

Fig (15 Directions of Vectors Specified with Respect to Geographical Directions

In this case also, *we* need to assign positive and negative signs, using Fig (13).

(iv) Sometime, the sequence of geographic directions is switched. In this case the given angle is the complementary angle of the actual angle. If, in the above example, all geographic directions were to be switched, we shall see something as follows:

F_1 at 57°, east of north, F_2 at 50°, west of north,

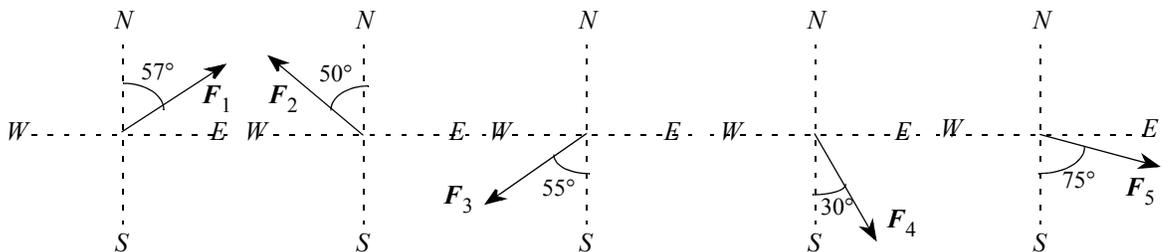F_3 at 55°, west of south F_4 at 30°, east of south,

F_5 at 75°, east of south.

Lengths of the arrows do not reflect their magnitudes; which, at this point, is unspecified.

Fig (16) Geographical Directions of Vectors Switched

In this case x-components will become the *opposite* sides of the right angled triangle and must be calculated as $sin\theta$. For example, $F_{1x} = F_1 sin 57$; and $F_{4y} = F_4 cos 30$. Alternatively, we may subtract the angle from 90° and use the usual procedure.

(v) Directly: Values of x- and y-components are given in "parentheses format".

For example: Vector **A:** (32, –47) means that A_x = 32 units and that A_y = –47 units. No calculations are needed. This is shown i Fig (16).
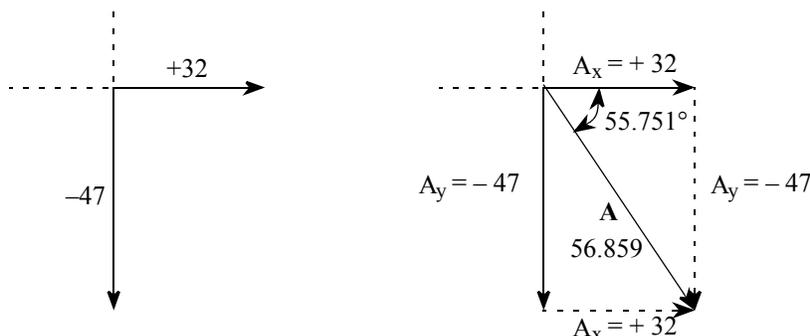
Fig ª17) Vector Given in Components Form

It is possible to reconstruct the vector from its components, using the Pythagorean Theorem. The magnitude is

$$A = \sqrt{(32)^2 + (47)^2} = \textbf{56.859} \text{ units} \qquad \theta = atan\left(\frac{47}{32}\right) = \textbf{55.751°} \quad \textbf{4th quadrant}$$

If we are combining vectors, we will not reconstruct the vector. If we are not combining vectors, we may be asked to reconstruct the vector, purely as an exercise.

10.12 A Format for Using the Analytical Technique

Let 5 forces act on an object. We shall combine the 5 force vectors and find the resultant force F_{net}. The unit of force is *Newton* and we write *N* for it. Let the forces be:

(i) 33 N at 33°with respect to positive x-axis, (ii) 40 N at 50° west of north,

(iii) 35 N at 35°, 3rd quadrant, (iv) 60 N at 60 south of east, and (v) (15, –4)

Vectors drawn to scale

Fig (18) Vectors to be Combined; Drawn to Scale

The vector equation is:

$$F_{net} = F_1 + F_2 + F_3 + F_4 + F_5$$

We shall make a table for the x- and y-components, as shown below. All numbers in the x-components column will be added scalerly, using a calculator. It is obvious that numbers carrying negative signs will get subtracted. The sum of these numbers will be the x-component of the resultant vector or (in this case) $F_{net,x}$. Similarly all numbers in the y-components column, will be added scalerly to yield $F_{net,y}$
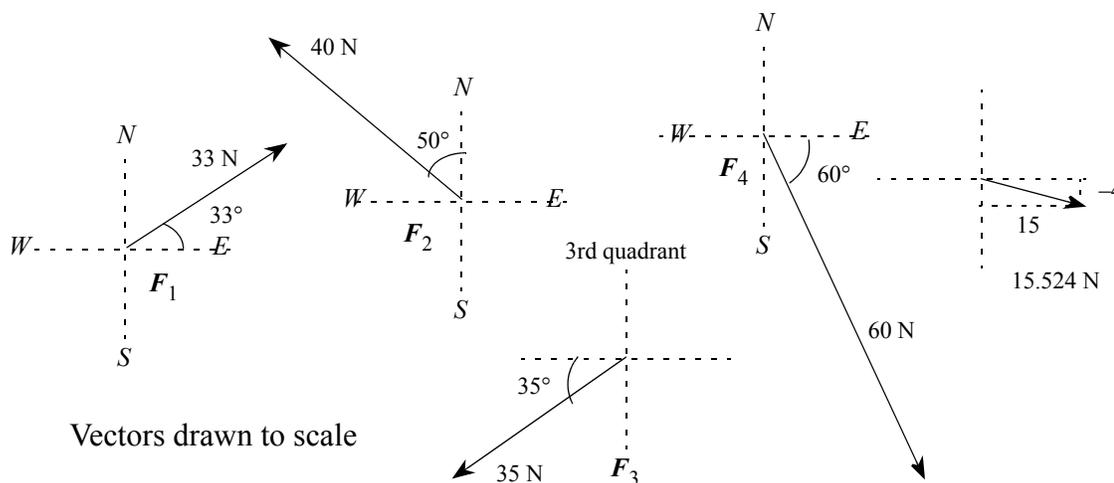
It is easy to see that if a vector per chance happens to be along the x-axis, then its y-component will be zero. The x-component, likewise, of a vector that per chance lies along the y-axis, will be zero.

Table 1: Combining Vectors Analytically

	Vectors	x-components	y-components
1	33 N, at 33° with x-axis	33 cos 33 = = + 27.676	33 sin 33 = + 17.973
2	40 N, 50° west of south or 40 N at 40° south of west	– 40 sin 50 = – 30.642 or – 40 cos (90–50) = – 30.642	+ 40 cos 50 = + 25.712 or + 40 sin (90–50) = + 25.712
3	35 N at 35°, 3rd quadrant	– 35 cos 35 = – 28.670	– 35 sin 35 = – 20.075
4	60 N at 60° south of east	+ 60 cos 60 = + 30	– 60 sin 60 = – 51.962
5	(15, –4)	+15	–4
	Adding numbers in each column scalerly	$F_{net,x}$ = + 13.364	$F_{net,y}$ = – 32.352

The resultant force F_{net}, lies in the fourth quadrant. Its magnitude and direction are found as:

$$F_{net} = \sqrt{(13.364)^2 + (32.352)^2} = 35.0035 \text{ N}$$

at

$$\theta = atan\left(\frac{32.352}{13.364}\right) = 67.555 °, \text{4th quadrant}$$

It is very important to monitor the algebraic signs of R_x and R_y. Fig (14) should then be used to determine the quadrant of the resultant vector. In fact, once we have determined the quadrant, the algebraic signs better be discarded. They will only lead to confusion. Mind you a negative R_x or a negative R_y will result in the same negative angle. A negative angle quoted as an answer, makes the answer indeterminate. The instructor will not know whether the resultant vector that you found, is in the second quadrant or in the fourth quadrant. Similarly if both R_x and R_y are negative, you will get a positive angle which will indicate that the resultant vector is in the first quadrant. No instructor will accept an indeterminate answer.

While inserting R_x and R_y in the equation:

$$R = \sqrt{R_x^2 + R_y^2}$$

the use of algebraic signs is a perfect waste of time and energy. Hence it is important to realize that the *only* function of the algebraic signs in R_x and R_y. is to specify the quadrant of the resultant vector. Having determined the quadrant, the signs should better be discarded.

10.13   Subtraction of vectors (1)

On occasions, in physics, we need to determine a change in the status of an object. Changes are always calculated as

change = (final status) – (initial status)

A change in the *displacement* status of an object will be written as $\Delta x = x_f - x_i$; a change in the *velocity* status of the object will be written as $\Delta v = v_f - v_i$, and so on. For example, if a car is travelling initially with a velocity v_o and then speeds up (or slows down) to a velocity v, we would like to know by how much its velocity changed. Or, consider a big tree trunk which is being pulled by some people in some direction, with a net force F_{net} . They then change the direction of pull (without changing the magnitude of the force). We would like to know by how much did the vector pulling force change. To find *changes*, we need to subtract one vector from another. Speaking purely from the *vectors* point of view however, we are sad to tell you that in vector algebra there is, unfortunately, no mechanism for subtracting vectors. An instruction of the type $(A - B)$ cannot be executed. Presently, we make the following explicit statement;

There is no such thing as *subtraction* of vectors.

How do we find the difference of two vectors? How do we subtract a vector?

10.14   Negative Vectors

The concept of negative vector stems from our ability of undoing things. If we take a step forward, we can take it back. We drive a car forward and we can drive it in reverse gear. We drive a nail in the wall and we can pull it out. Undoing is the negation of doing.

As vectors in physics always do something on an object, they can undo as well. A vector that *undoes* things for us, is called a *negative* vector. It should be carefully noted that in order to have a negative vector, we have to have a vector first. We cannot have a negative vector out of nowhere. Do not let anybody tell you that since the vector is directed along the negative x-axis (for example), therefore it is a negative vector. Because it is *not*!

In order to undo something, we act in the opposite direction. The vector that undoes something for us, therefore, must act in the opposite direction to the one that is doing that thing. Hence a negative vector is one that is directed opposite to the original vector. We define the negative of a vector as the vector that has gotten its tip and toe switched around. *The magnitude of the negative vector doesn't become negative.* Only orientation changes by 180°. The negative vector of vector A is written as $-A$. The magnitudes of A and $-A$ are identical and *necessarily* positive. It is always possible to form the negative vector of a given vector. Following diagram shows some vectors and their negative vectors.

Fig (18) Negative Vectors of Given Vectors
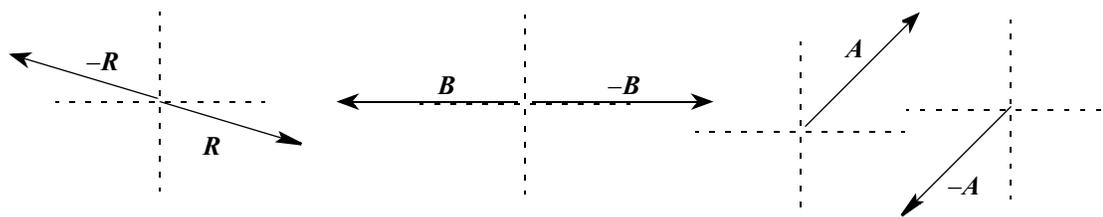
10.15   Subtraction of vectors (2)

Now that we can convert a vector into a negative vector, it is possible to "subtract" one vector from another vector. All we need do is to form the negative vector of the vector to be subtracted and then use any of the combination techniques to combine the two. The end result will be the difference of the two vectors. Thus

$$A - B = A + (-B)$$

An Example:

A car is travelling at 35 km/h in a direction 27° north of east. The road bends. The car is then found to be travelling at 50 km/h in a direction 72° north of east. By how much did the velocity of the car change?

The change in the velocity of the car, Δv is to be found as:
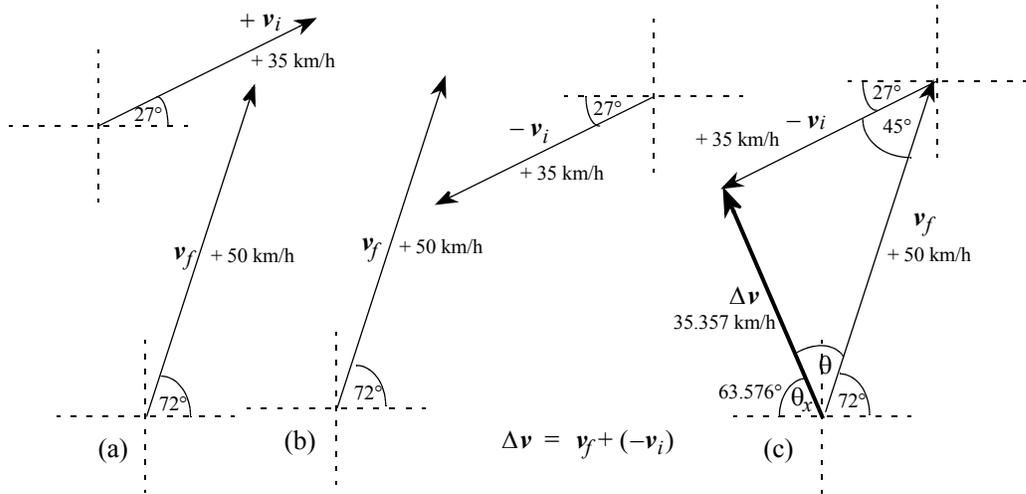
$$\Delta v = (v_f - v_i) = v_f + (-v_i)$$

$$\Delta v = v_f + (-v_i)$$

Fig (19) Finding Δv

Fig (19a) shows the two vectors v_i, and v_f. Fig (19b) shows vectors $-v_i$ and v_f,. In Fig (19c), the vectors v_f and $-v_i$ have been tiptoed (the standard procedure for *combining* vectors) to form vector Δv. The magnitude and direction of Δv can be found either by using the geometrical method or the analytical method. Both are shown below.

For the geometrical solution, we find that the angle opposite side Δv is $(72° - 27°) = 45°$. The cosine law gives us:

$$\Delta v = \sqrt{(50)^2 + (35)^2 - (2)(50)(35)(cos\,45)} = 35.357 \text{ km/h}$$

and

$$\frac{sin\,45}{35.357} = \frac{sin\,\theta}{35} \qquad \theta = 44.424° \qquad \theta_x = 180 - 72 - 44.424 = 63.576°$$

For the analytical solution, we form the table of x- and y-components.

Table 2: Combining Vectors Analytically

	Vectors	x-components	y-components
1	50 km/h at 72°, north of east	+ 50 cos 72 = + 15.451	+ 50 sin72 = + 47.553
2	35 km/h, at 27° south of west	− 35 cos 27 = − 31.185	− 35 sin 27 = − 15.890
	Adding numbers in each column scalerly	$(\Delta v)_x = -15.734$	$(\Delta v)_y = +31.663$

The resultant vector Δv lies in the second quadrant. Its magnitude and direction are found as:

$$\Delta v = \sqrt{(15.734)^2 + (31.663)^2} = 35.357 \text{ km/h}$$

at

$$\theta = atan\left(\frac{31.663}{15.734}\right) = 63.576° \text{ , 2nd quadrant}$$

It should be noted that the geometrical and the analytical methods have produced identical results. It should further be noted that if we were given the speeds (and not the velocities), we wouldn't have done any of the above. The change in speed simply would have been $(50 - 35) = 15$ km/h! Does it surprise you to see that you were travelling with a velocity of 35 km/h and you changed the velocity by about 35 km/h and the resultant velocity is only 50 km/h?

To illustrate this point clearly, consider:

$$\Delta v = v_f - v_i$$

rearranging

$$\Delta v + v_i = v_f$$

This is shown in Fig (20) where we switched $-v_i$ of Fig (19c), back to $+v_i$ and tiptoed it to the vector Δv to obtain the left hand side of the last equation. Now v_f appears as the resultant vector.
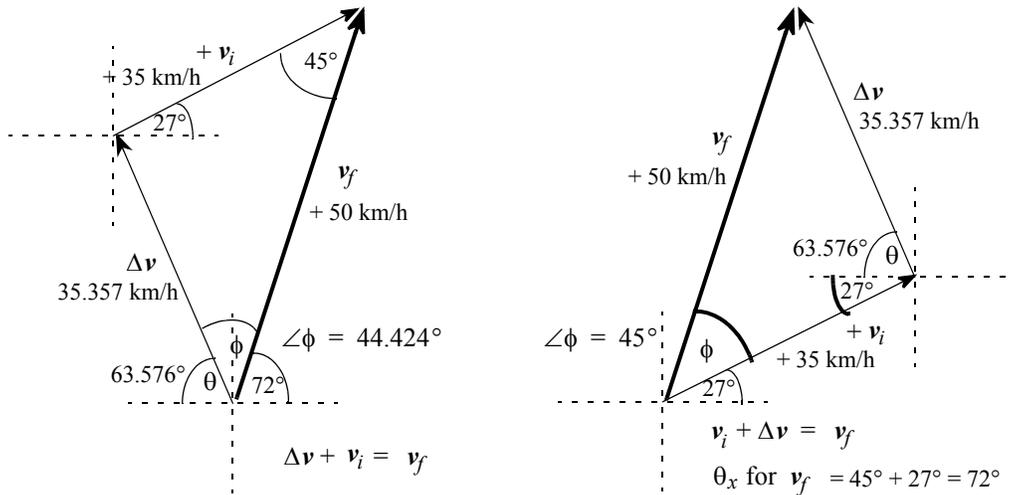
Fig (20) Rearranging Eqn $\Delta v = v_f - v_i$ in Two Different Ways.

For each of the two diagrams of Fig (20), the magnitude of v_f is found by using the cosine law (comes out to be 50 km/h) and the angles ϕ is found by using the sine law (comes out to be $45°$). The calculations are left to the student as an exercise.

Another Example

Suppose some guys are pulling a tree trunk with a net force F_{net}, eastward. They then change the direction of pull by $60°$, keeping the magnitude of the force constant. The force in the new direction is called F'_{net}. Following diagram shows the vectors F_{net} and F'_{net}. Follow the sequence of operations to find $\Delta F_{net} = F'_{net} - F_{net}$
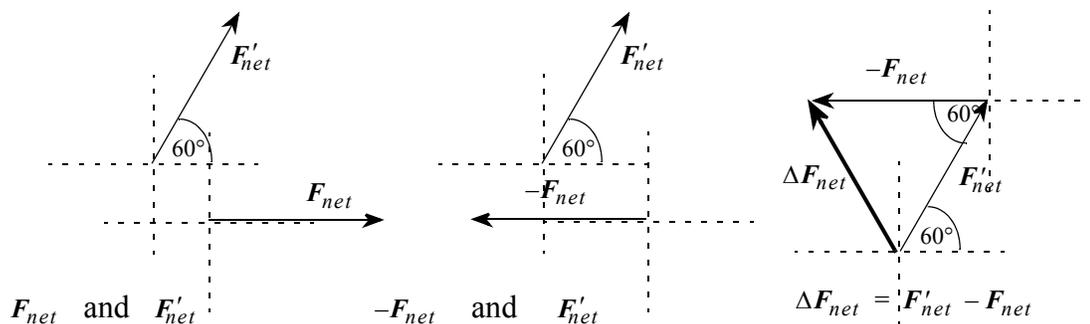
Fig (21) Finding ΔF_{net}

The geometrical method, in this example will be found to be far simpler and easier. No calculations are needed. The triangle is an equilateral triangle. The magnitudes of all three vectors are equal:

$$\Delta \boldsymbol{F}_{net} = \boldsymbol{F}_{net} = \boldsymbol{F}'_{net}$$

The magnitude of vector representing the change in net force equals the magnitude of the original net force and lies at 60°, (quite visibly) in the second quadrant.

Let's try the analytical method. Please note that the magnitudes of vectors $\boldsymbol{F}_{net}$ and $\boldsymbol{F}'_{net}$ is the same. Thus $F'_{net} = F_{net}$.

### Table 3: Combining Vectors Analytically

	Vectors	x-components	y-components
1	$\boldsymbol{F}'_{net}$, 60°, 1st quadrant	$+ F'_{net} \cos 60 = + 0.50\ F'_{net}$ $= + 0.50\ F_{net}$	$+ F'_{net} \sin 60 = + 0.866\ F'_{net}$ $= + 0.866\ F_{net}$
2	$\boldsymbol{F}_{net}$, directed west	$- F_{net}$	0.00
	Adding numbers in each column scalerly	$- 0.50\ F_{net}$	$+ 0.866\ F_{net}$

The resultant vector $\Delta \boldsymbol{F}_{net}$, lies in the second quadrant. Its magnitude and direction are found as:

$$\Delta F_{net} = \sqrt{(0.50 F_{net})^2 + ((0.866) F_{net})^2} = F_{net}$$

at

$$\theta = atan\left(\frac{0.866}{0.50}\right) = 60.000\ °,\ \text{2nd quadrant}$$

Does it surprise you that the magnitude of all three vectors is the same? You added F_{net} to: F_{net} and the result is F_{net} . If $F_{net} = 2$ units , we would be writing (2 + 2 = 2).

10.16 Negative Vectors and the Analytical Method

In the analytical method the switching of tip and toe translates into sending the vector into the opposite quadrant. This, in turn, results in getting the algebraic signs of components switched. If a component had positive sign, it will become negative and *vice versa*.

Let us recall an earlier example where we combined five vectors as:

$$\boldsymbol{F}_{net} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_4 + \boldsymbol{F}_5$$

We shall now work out the following combination of the same five vectors:

$$\boldsymbol{F}_{net} = (-\boldsymbol{F}_1) + \boldsymbol{F}_2 + (-\boldsymbol{F}_3) + \boldsymbol{F}_4 + (-\boldsymbol{F}_5)$$

It can also be written as:

$$\boldsymbol{F}_{net} = -\boldsymbol{F}_1 + \boldsymbol{F}_2 - \boldsymbol{F}_3 + \boldsymbol{F}_4 - \boldsymbol{F}_5$$

The original vectors are;

(i) 33 N at 33°with respect to positive x-axis, (ii) 40 N at 50° west of north,

(iii) 35 N at 35°, 3rd quadrant, (iv) 60 N at 60 south of east, and (v) (15, –4)

The vectors to be combined are:

(i) 33 N at 33°with respect to **negative** x-axis, (ii) 40 N at 50° west of north,

(iii) 35 N at 35°,***1st*** quadrant, (iv) 60 N at 60 south of east, and (v) ***(–15, +4)***

Table 4: Combining Vectors Analytically

	Vectors	x-components	y-components
1	33 N, at 33° with –x-axis	– 33 cos 33 = = – 27.676	– 33 sin 33 = – 17.973
2	40 N, 50° west of south or 40 N at 40° south of west	– 40 sin 50 = – 30.642 or – 40 cos (90–50) = – 30.642	+ 40 cos 50 = + 25.712 or + 40 sin (90–50) = + 25.712
3	35 N at 35°, 1st quadrant	+ 35 cos 35 = + 28.670	+ 35 sin 35 = + 20.075
4	60 N at 60° south of east	+ 60 cos 60 = + 30	– 60 sin 60 = – 51.962
5	(15, –4)	– 15	+ 4
	Adding numbers in each column scalerly	$F_{net, x}$ = – 15.352	$F_{net, y}$ = – 20.148

The resultant force F_{net} , lies in the third quadrant. Its magnitude and direction are found as:

$$F_{net} = \sqrt{(13.364)^2 + (32.352)^2} = 25.330 \ \text{N}$$

at

$$\theta = atan\left(\frac{20.148}{15.352}\right) = 52.694 \ °, \text{3rd quadrant}$$

10.17 The Role of the Resultant Vector in Physics

Having familiarized ourselves with the concept of resultant vector and having studied several examples of finding the resultant vector of a set of given vectors, we shall now talk about the role of resultant vectors in physics.

The Resultant Vector

Because of being opposite in orientation to other vectors, we can easily recognize the resultant vector in a polygon of vectors. It is important to realize that a *resultant* vector represents all other vectors. By itself, it will do *everything* that all the other vectors *together* would have done. It makes the original set of vectors redundant.

Essentially, Only One Vector

In physics if a set of vectors, acting on an object, influence the object, and produce a certain effect then the resultant vector acting alone on that object, will produce exactly the same effect. Because of this, for computational purposes, it is necessary to replace the original set of vectors by their resultant vector and then conveniently *forget* the original set of vectors. One deals, essentially, with only *one* vector, instead of dealing simultaneously with a multitude of vectors.

Something is Happening

(a) Vectors **A** and **B** act on the object

(b) **A** and **B** are tiptoed and replaced by **R**
Only **R** acts on the object

(c) All three vectors are shown attached to the object. Either **R** acts or **A** &**B** act on the object, but not all three.
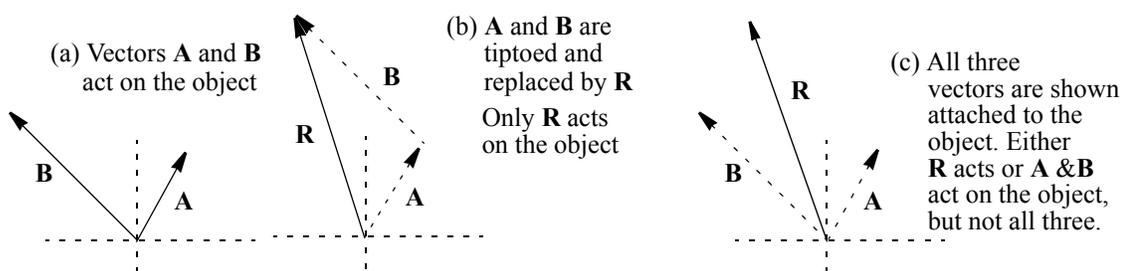
Fig (22) Two Vectors; their Resultant Vector

Whenever an object is being influenced by a resultant vector, something must be happening to it and hence it must be undergoing a change of state. As such it cannot be in a stable or equilibrium state and must, therefore, be found in a activity region. We conclude:

Resultant vectors are found in Activity Regions only.

An Important Conclusion

Without discussion, we shall state the following:

Only Resultant Vectors can cause things to happen to a physical system

This is because only *resultant vectors* are found in the activity regions.

10.18 The Equilibriant Vector

An important outcome of our ability to form the negative vector of a given vector is to form the negative vector of the *resultant* vector. It is a vector of fundamental importance and has a special name. It is called the **equilibriant** vector. It is formed by switching around the tip and the toe of the resultant vector. We write E for it and, by definition

$$E = -R \qquad \qquad ....(6)$$

10.19 The Role of the Equilibriant Vector in Physics

Because the equilibriant vector is a *tiptoe* combination (like others), it is impossible to recognize the equilibriant vector in a polygon of vectors. Quite like the resultant vector, the equilibriant vector is not one of the original set of vectors. Hence the equilibriant vector is also an outsider; but it becomes an indistinguishable member of the group. Unlike the resultant vector that represents the original set of vectors and replaces them all, the equilibriant vector does not represent or replace other vectors. It *infiltrates* the group and becomes one of them. The *gang* (consisting of all the original vectors **plus** the infiltrator) then *conspires* to do nothing, *absolutely nothing*!
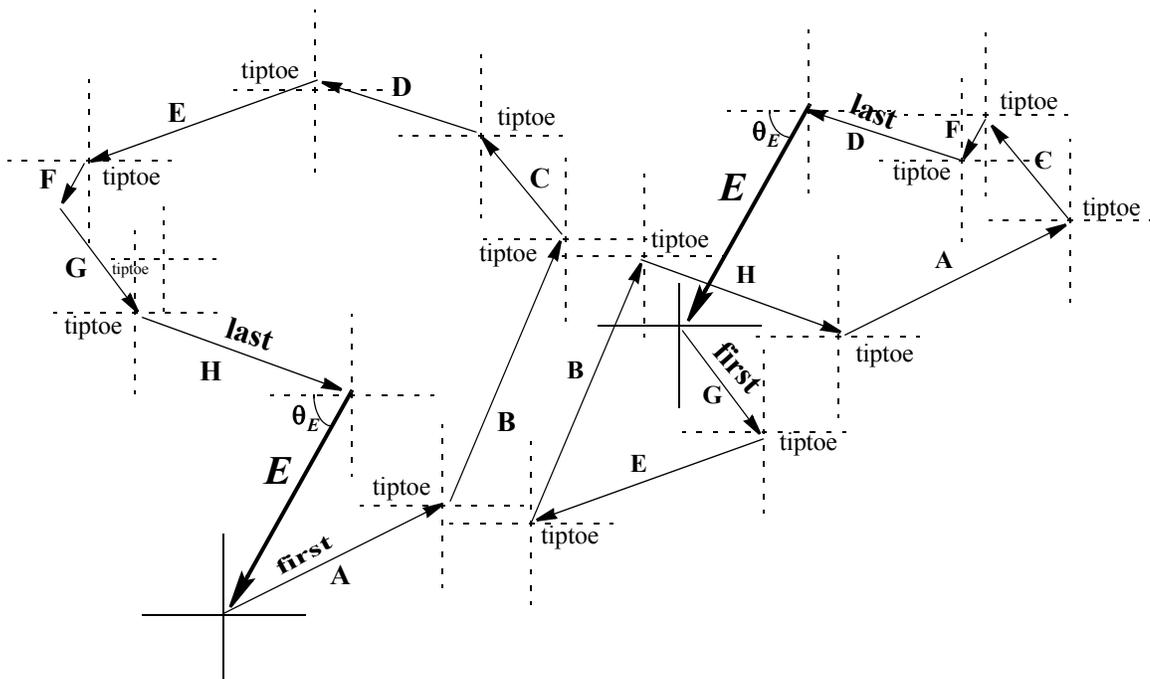

Fig (23) Converting Resultant Vector into Equilibriant Vector

Fig (23) shows the vector polygon of Fig (4), scaled down to 85%, where we have replaced the resultant vector by the equilibriant vector.

The angle and the reference frame have also gotten repositioned. We would like to point out that unless you notice the letter *E* in italics, you may never know which vector is the equilibriant vector. *Any of the multitude of vectors could be the equilibriant vector.*

Essentially Any Vector

How about the tiniest of them all, the **F**-vector? It will be correct to say that this tiny vector has successfully neutralized the effect of 8 of its big companions. It may appear to you to be somewhat like the youngest of nine brother who successfully counters the efforts of 8 of his big brothers! Please note that when it comes to equilibrium, any vector is as good an equilibriant as another. We make no distinction between the vectors and *we do not* wish to know which of the participant vectors actually brought the system to equilibrium. If the statement of a problem indicates that such and such vector is the equilibriant vector, we shall just use the information to complete the polygon.

Nothing is Happening

In physics if a set of vectors forming a polygon, is acting upon an object, the object does not get affected by *any or all* of those vectors! It is as if the entire set of vectors did not exist! We may, very conveniently, ignore them all completely. As such, we shall find them in equilibrium states only. The apparent polygon of vectors shown in Fig (4) is really not a polygon because the resultant vector *R* has annihilated the other 8 vectors.
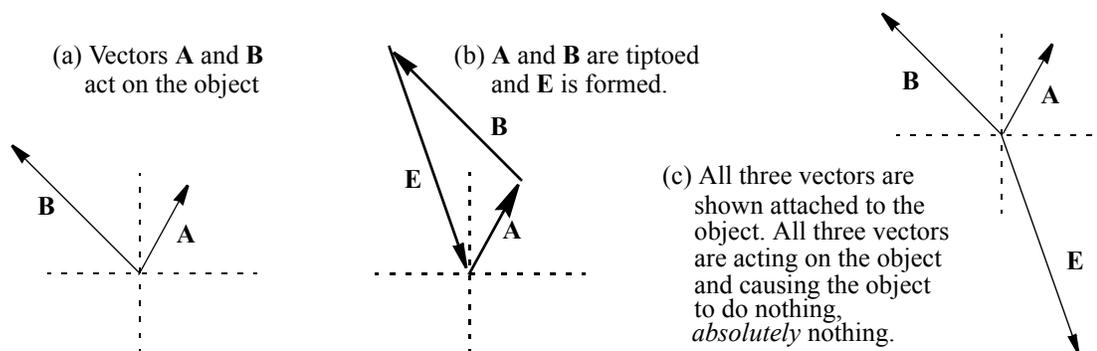

(a) Vectors **A** and **B** act on the object

(b) **A** and **B** are tiptoed and **E** is formed.

(c) All three vectors are shown attached to the object. All three vectors are acting on the object and causing the object to do nothing, *absolutely* nothing.

Fig (24) Two Vectors; their Equilibriant Vector

We make the following statement:

Polygons of vectors are found in Equilibrium States only. Conversely, if an object is in an equilibrium state, the vectors present must necessarily form a polygon.

Equilibriant Vector as a Freezing Agent

Because the equilibriant vector neutralizes the effect of an aggregate of vectors that act on an object, the object is rendered immune to the effect of those vectors. The equilibriant vector may, therefore, be viewed as a *freezing agent.* The freezing agent serves to *freeze* or *incapacitate* or *immobilize* the object in a state of equilibrium, where it will be either at rest or in uniform motion. The concept is extremely important; especially in case of force vectors.

Another Important Conclusion

Only Equilibriant Vectors can cause things
NOT to happen to a physical system

A Table Comparing Resultant and Equilibriant Vectors:

Table 5: A Group of (*n>1*) Vectors is Acting on an Object

Resultant Vector	Equilibriant Vector
It is an *outside* or *additional* vector. Doesn't belong to the group	
the original group of vectors is annihilated	the original group of vectors remains intact
total number of vectors: = *1*	total number of vectors = *(n+1)*
essentially only one vector acts on the object	essentially all *(n+1)* vectors act on the object
causes things *to happen* to the object	causes things *not to happen* to the object
found in the *activity regions* only	found in the *equilibrium states* only
if *(n = 1)*, then this vector is a resultant vector	to have an equilibriant vector, *n* must be greater than one

nothing is happening here

another equilibrium state

something is happening here

Activity Region

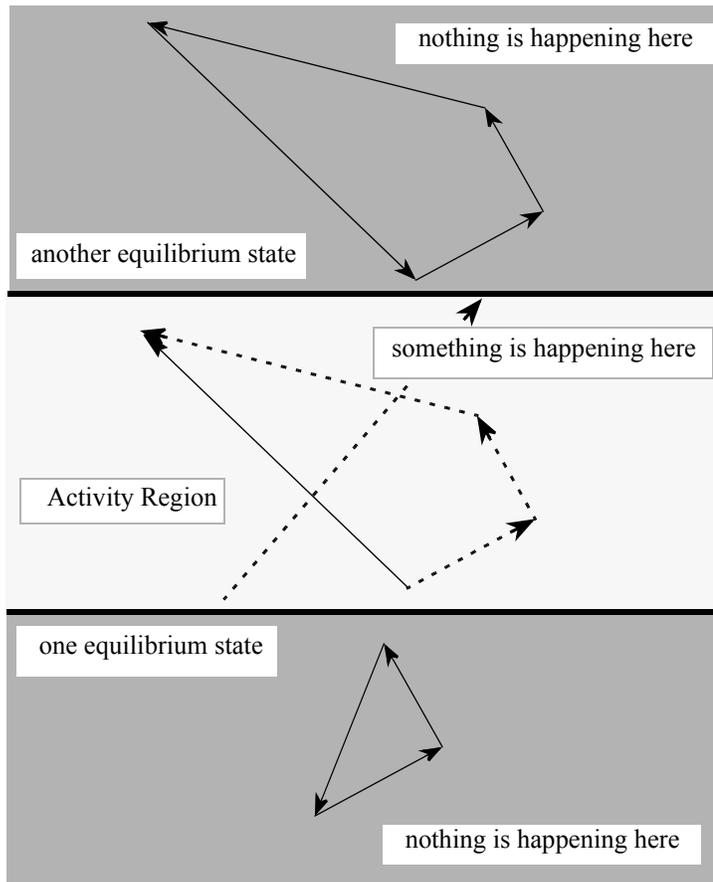one equilibrium state

nothing is happening here

Fig (25) Habitats of Open-Ended & Closed-Ended Configurations of Vectors.

10.20 Multiplication of Vectors: (a) Multiplying a Vector by a Scaler

Multiple occurrences of an event are represented by the mathematical operation "multiplication". Five visits to the grocery store may be written as (5) x (visit). If the event is to be represented by a vector, such as A, we shall write $5A$. The quantity $5A$ will be a vector quantity even though 5 is scaler. We are said to have *multiplied* a vector by a scaler.

The result of this multiplication is a vector. Weight is a good example. We multiply the scaler mass *m* of the object with the vector acceleration due to earth's pull *g* and get the weight vector *mg* . Momentum *mv* and *at* are some other examples.

As another variation of the theme, you may be asked to combine vectors $3.5A$, $17B$, xC , and so on.

10.21 Multiplication of Vectors (b) Multiplying a Vector with another Vector

This is rather complicated and is really beyond the scope of this textbook. It will, however, be in order to give you some basic information without going into detailed discussions. Vectors can be multiplied to one another in two different ways. These are called (i) dot multiplication, and (ii) cross multiplication. The result of dot multiplication is a scaler quantity. When multiplied in the *dot* manner, the vectors get totally *dis-oriented*. In particular we write:

$$A \bullet B = (A)(b)(cos\theta) = C \qquad\qquad ....(7)$$

where $\angle\theta$ is the angle between the two vectors. Examples of such vector multiplication in physics are work ($F \bullet d$) and power. ($f \bullet v$). In case of a cross product, the result of the multiplication is still a vector. A cross product of two vectors is written and defined as

$$A \times B = (A)(B)(sin\theta)n = C \qquad\qquad ....(8)$$

The direction $\hat{n}$ of the vector formed as a result of multiplication, is perpendicular simultaneously to both the original vectors. Thus, if *A* and *B* are in the (x,y) plane, the product vector *C* will be along the z-axis. Examples of such vector multiplication in physics are

    (i) torque $\qquad\qquad\qquad \tau = F \times d = (F)(d)(sin\theta)n \quad$,

and

    (ii) the magnetic force $\qquad F_B = (q)(v)(B)(sin\theta)\,n \quad$.

10.22 Squaring a Vector

This is a special case of the scaler product of two vectors. Suppose we dot multiply the vector by itself, we shall have to set *B* = *A* in Eqn (7) above. Also the angle between the two *A* vectors will be zero degrees. We shall get:

$$A \bullet A = (A)(A)(cos0°) = A^2$$

We find that by squaring the vector, we got a scaler entity, a pure number! Supposing we cross multiplied the vector by itself, we shall get from Eqn (8):

$$A \times A = (A)(A)(sin0°) = 0$$

We find that there is only one way of multiplying a vector by itself (i.e. squaring it). The result is a scaler number. Sometimes it pays to use squares of vectors rather than vectors themselves.

10.23 Solving Problems Using Vector Geometry

The vector geometrical method of solving problems is very important in physics. In this book, we shall use the vector geometrical method in preference to the analytical method in practically all cases where three vectors form a triangle, and occasionally in cases where four vectors form a quadrilateral.

10.24 *Three Vectors Forming a Triangle*

A set of three vectors that form a triangle, has several interesting applications in physics. Often times we are given two vectors and we need to find the resultant vector or the equilibriant vector. The three vectors, in such cases, make a triangle. Again, if an object is

being acted upon by three forces and the object is in equilibrium, the three vectors must make a triangle. Relative velocity and conservation of momentum are interesting areas where three vectors form a triangle. The details of the method are given earlier in the section "Combining just-any-two vectors". We shall illustrate the usefulness of the method with the help of two examples.

Sometime we may give you four vectors with two being along the same straight line. In this case these two vectors will be converted into one vector using rules #1 or #2. This will leave us with three vectors.

First Example

Vectors *A*, *B* and *C* act on an object whose center of mass lies at the origin of our reference frame and cause it to be in equilibrium. Vector *A* lies at 33° in the first quadrant. Vector *B* lies at 40° in the second quadrant. Vector *C* whose magnitude is 137 units is directed south. Find the magnitudes of vectors *A* and *B*.

Fig (26a) below, shows vectors *A*, *B* and *C*, while Fig (26b) shows the triangle. We start with vector *A* in the first quadrant and choose its length quite arbitrarily. At the tip of vector *A*, we place another reference frame and draw vector *B* (second quadrant). The length of vector *B* is also unknown. As the third vector lies on the y-axis, and the three vectors must form a triangle, we keep extending the length of vector *B* until it hits the y-axis. This completes the triangle

In this particular example, we shall use the sine law alone and successfully obtain the magnitudes of vectors *A* and *B*.
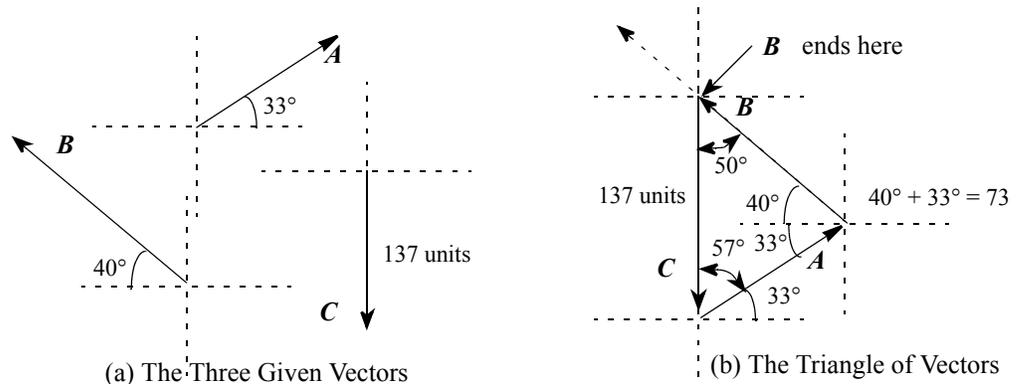

(a) The Three Given Vectors   (b) The Triangle of Vectors

Fig (26) Vectors & Their Triangle of the "First Example"

$$\frac{A}{\sin 50} = \frac{137}{\sin 73} = \frac{B}{\sin 57}$$

This gives us

$$A = \left(\frac{137}{\sin 73}\right)(\sin 50) = \textbf{109.743 units}$$

and

$$B = \left(\frac{137}{\sin 73}\right)(\sin 57) = \textbf{120.148 units}$$

Second Example

A compass needle points 18° east of north outdoors due to the earth's magnetic field whose strength is 50 µT (micro-tesla). Inside a building, there is a magnetic field of magnitude 33 µT at 75°, 4th quadrant. If the compass needle is brought inside the building, in which direction will it point? What is the magnitude of the net magnetic field inside the building?

Let the earth's magnetic field be named vector A and that inside the building be called vector B. We are then required, essentially, to find the resultant vector (magnitude and direction). Fig (27a) below, shows the two given vectors.

(a) The Given Vectors
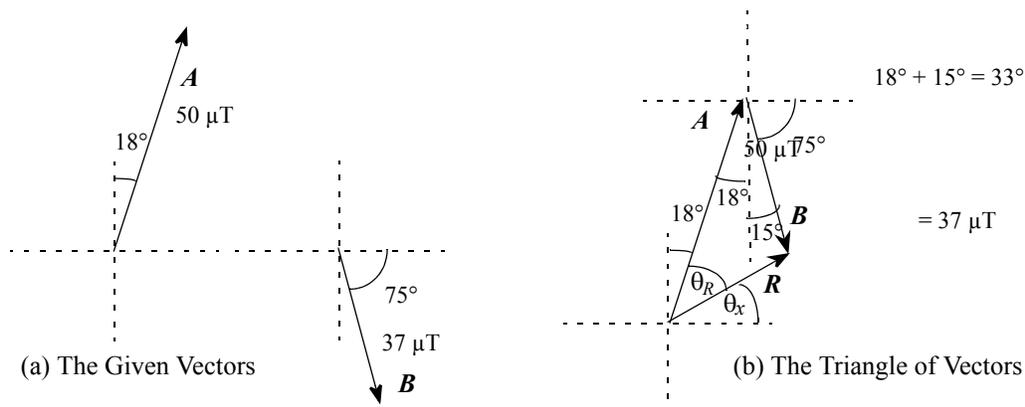
(b) The Triangle of Vectors

Fig (27) Vectors & Their Triangle of the "Second Example"

In Fig (27b) the two vectors are tiptoed and the resultant is drawn. Necessary angles have been calculated and shown.

As the angle opposite vector R is 33°, we use cosine law to find the magnitude of the resultant vector:

$$R = \sqrt{(50)^2 + (37)^2 - (2)(50)(37)(cos\,33)} = \textbf{27.675 μT}$$

Next, use sine law to find θ_R :

$$\frac{sin\,\theta_R}{37} = \frac{sin\,33}{27.675} \qquad \theta_R = 46.732°$$

The angle with respect to the x-axis, θ_x , is found as:

$$\theta_x = 90 - 46.732 - 18 = \textbf{25.268°}$$

This completes the answer.

10.25   *Four Vectors Forming a Quadrilateral*

Consider four vectors A, B, C and D. Since the vectors form a quadrilateral (closed configuration) the vector sum of the four vectors is zero.
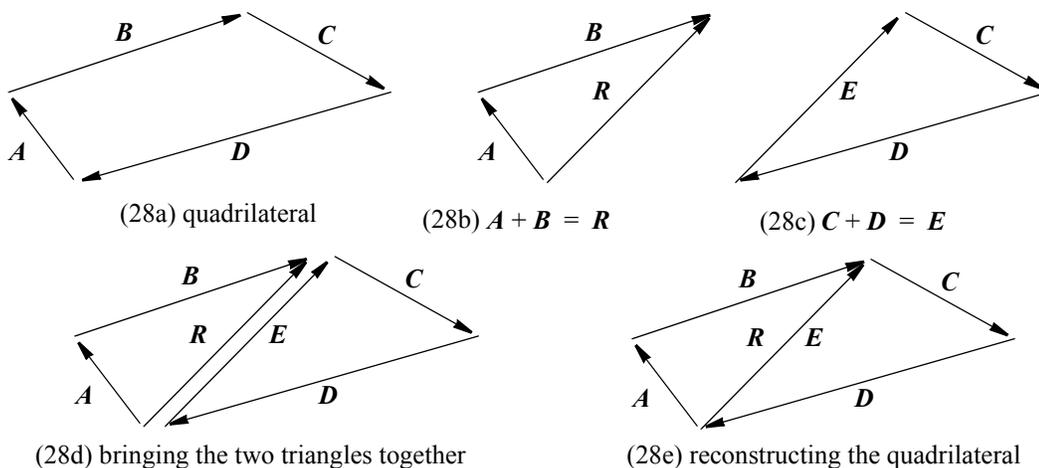

(28a) quadrilateral

(28b) $A + B = R$

(28c) $C + D = E$

(28d) bringing the two triangles together

(28e) reconstructing the quadrilateral

Fig (28) Splitting the Quadrilateral into Two Triangles

$$A + B + C + D = 0$$

This can be written as $\qquad A + B = -(C + D)$

Introducing a fifth vector, R, we write:

$$A + B = R = -(C + D)$$

where R is the resultant of vectors A and B. It is obvious that vectors A, B, and R will form a triangle.

$$A + B = R \qquad\qquad\qquad\qquad ....(9)$$

Again $R = -(C + D)$ can be written as $C + D = -R$. But as $-R = E$, we write:

$$C + D = E \qquad\qquad\qquad\qquad ....(10)$$

It is obvious that vectors C, D, and E will also make a triangle. The two triangles will have one common side. It will be R for one triangle and E for the other. This is shown in Fig (28) above.

From Eqns (9) & (10) we draw the following inference:

If four vectors make a quadrilateral, then the resultant of any two vectors (out of four) is the equilibriant of the remaining two vectors.

Some examples of the use of the above technique are: (i) three forces and their resultant or equilibriant, (ii) four forces, acting on an object, cause the object to be in equilibrium, (iii) three forces acting on a rotating object and causing it to be in radial equilibrium, and (iv) momentum conservation with two objects being independently in motion before collision and again being independently in motion after collision. All vectors are supposed to be in the same plane (co-planar).

We shall talk about these later on in the book.