Experiment #18

Equilibrium of a Crane

Principles

Equilibrium, Definition

The state of equilibrium of a system is defined as *state of rest* or state of *uniform motion.* The uniform motion may be in a straight line or in a circle. Thus an object or system of objects may travel in a straight line with uniform translational or (linear) velocity or it may travel in a circular path with a uniform rotational (or angular) velocity.

Equilibrium: Conditions

To be able to travel in a straight line with uniform motion, (or *not* to travel), the vector sum of all forces acting upon the system must be zero. Similarly, to be able to rotate in a circular path with uniform angular velocity, (or *not* to rotate), the sum of all torques acting upon the system must be zero:

$$\boldsymbol{F}_{net} = 0 \quad \text{or} \quad \Sigma \boldsymbol{F} = 0 \qquad \text{..........(1)}$$

$$\tau_{net} = 0 \qquad \Sigma\, \tau_{cw} = \Sigma \tau_{ccw} \qquad \text{..........(2)}$$

For complete equilibrium, therefore, it is necessary that both conditions be satisfied simultaneously. Or

$$\Sigma F = 0 \quad \text{and} \quad \Sigma \tau_{cw} = \Sigma \tau_{ccw} \qquad \text{..........(3)}$$

It is interesting to note that, from our list of four forces, F_P gets thrown out for obvious reasons We are left with three forces: F_T , F_C and F_g. It is, therefore, highly likely that one would end up forming a triangle (even though some of them may appear more than once) and apply sine and cosine laws. In case the system is one dimensional, the vectors will be treated as scalers.

Applying Conditions of Equilibrium

In principle, one should always be able to make a triangle of force vectors. But in systems involving F_C , one is unable to do so because of F_C 's rather conspicuous nature. A neat trick is to start the analysis with the torque equation by choosing the axis of rotation at the foot of the force F_C , thereby "intoxicating" it. One is then left with the easily manageable forces: F_T and F_g.

Some Examples

(a) The Contact Force is Conspicuously Absent

Consider a signboard suspended by two unneighborly cables as shown in Fig (1a). We shall calculate the tension forces F_{T1} and F_{T2} in these cables.

As cables are unneighbors, there is no F_C . We place the object (the signboard) at the origin of a reference frame and parallel-transport all 3 vectors, as shown in Fig (1b). Next we tip-toe the three vectors, as shown in Fig (1c). The three vectors must form a triangle because the system is in equilibrium. After writing down the angles, we apply the sine law:

$$\frac{F_{T1}}{\sin 53} = \frac{98}{\sin 82} = \frac{F_{T2}}{\sin 45} \qquad \text{..........(4)}$$

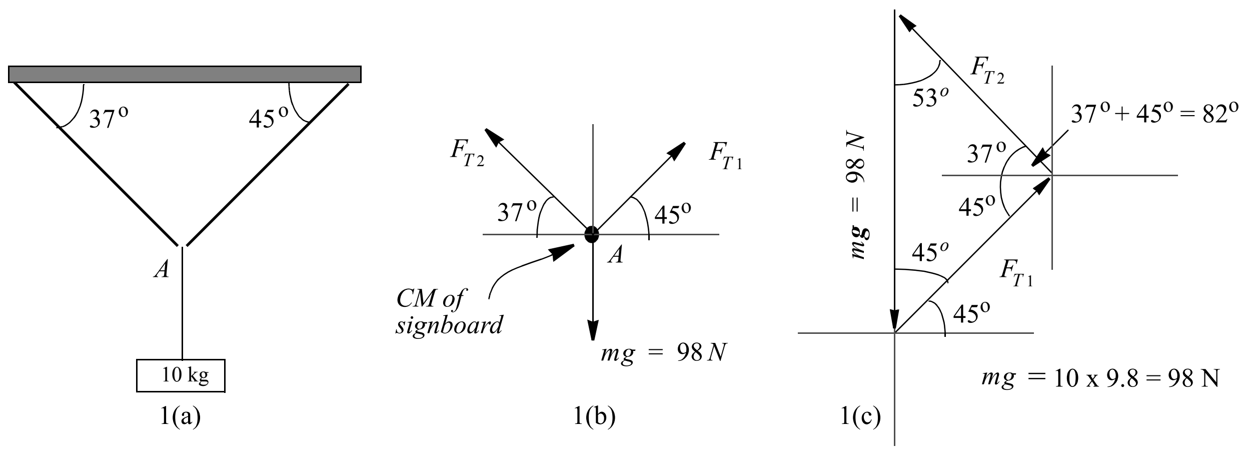

Fig (1) All three forces are acting at point 'A'

Solving for F_{T1} and F_{T2}, we find:

$$F_{T1} = 79.035 \text{ N; and } F_{T2} = 69.977 \text{ N} \qquad \text{..........(5)}$$

This completes the solution.

(b) The Contact Force is Inconspicuously Present

Consider a crane. A crane is used to lift loads vertically upward with uniform speed. Thus it is always in equilibrium. A static diagram that represents a freeze-up of the motion of the crane, is shown in Fig (2). The crane consists of a boom, pivoted at A on a vertical support. The boom is further supported by an unneighborly cable tied to the same vertical support at B. The mass of the load is M_L kg and that of the boom itself is M kg. The boom is of uniform dimensions and as such, its center-of-mass (CM) lies at its geometrical center. The diagram shows all forces and other relevant data. One would like to calculate the tension force F_T in the cable and the contact force, F_C , exerted by the vertical support on the boom.

As F_C is involved, we start with the torque equation and choose an axis at A, the foot of the force F_C , thereby rendering its torque zero. We find that the torque of the tension force F_T balance the torques of the two weight forces $M\boldsymbol{g}$ and $M_L\boldsymbol{g}$:

$$\tau_T = \tau_{Mg} + \tau_{M_L g} \qquad \text{..........(6)}$$

The torques are:

$$\tau_T = (F_T)(l)(\sin\theta_2) \qquad \tau_{mg} = (Mg)\left(\frac{l}{2}\right)(\sin\theta_1) \qquad \tau_{Mg} = (M_L g)(l)(\sin\theta_1) \qquad \text{..........(7)}$$

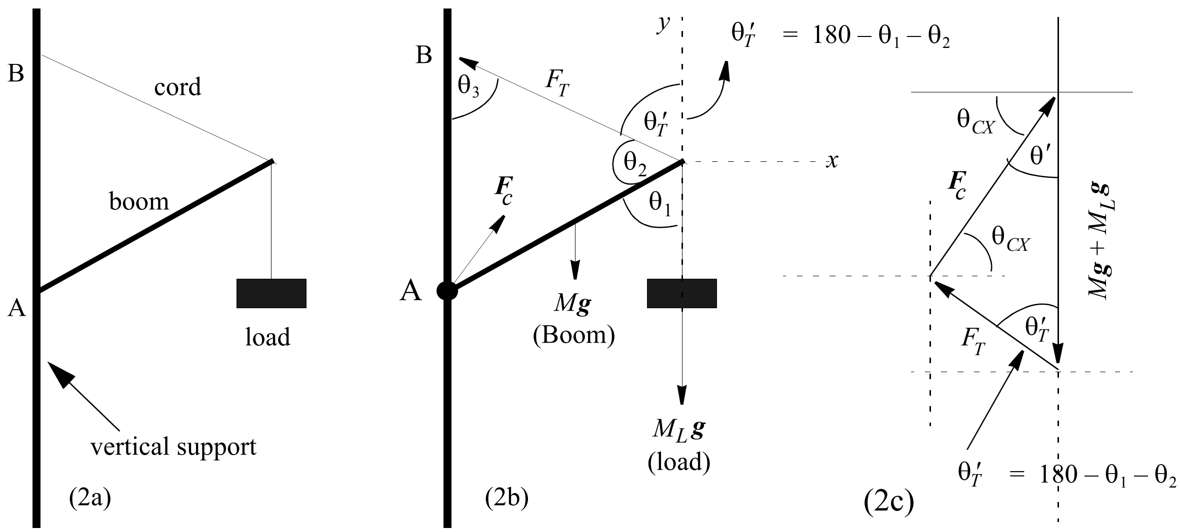

Fig (2) A Crane in Equilibrium

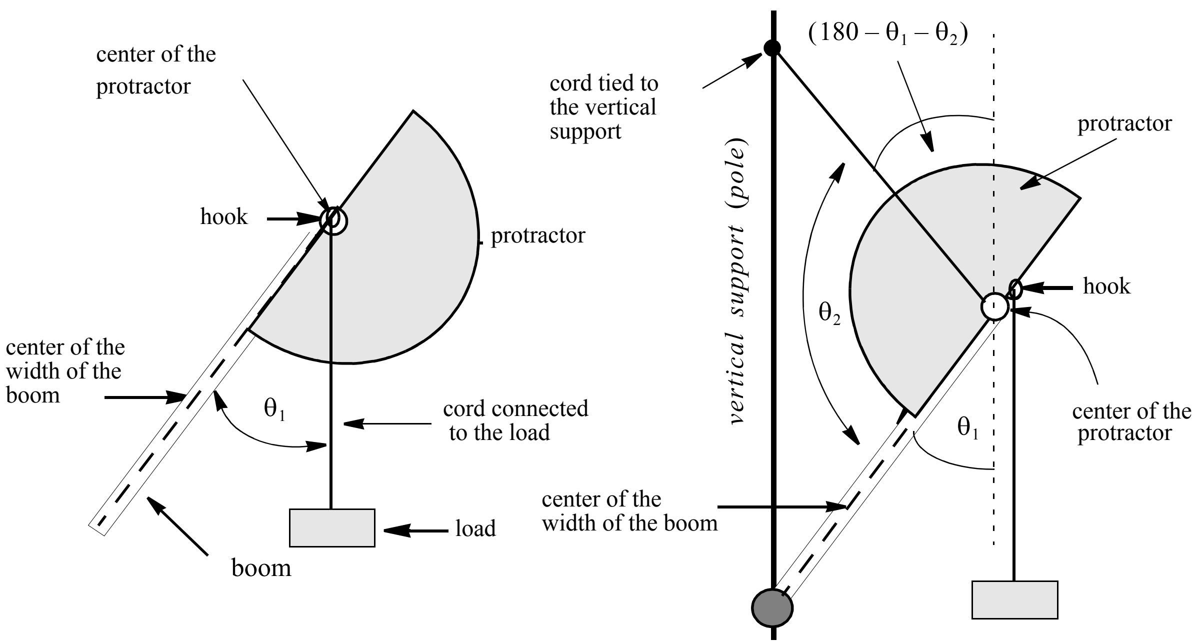

Fig (3) Measurement of Angles θ_1 and θ_2 for the Crane

To measure angles θ_1 and θ_2 we use a protractor as shown in Fig (3). From Eqns (6) and (7) we find:

$$F_T = \left(\frac{Mg}{2} + M_L g\right)\left(\frac{\sin\theta_1}{\sin\theta_2}\right) \qquad ....(8)$$

The angle θ_T' that F_T makes with the z-axis (and not with the x-axis), is easily seen to be $\theta_T' = 180 - (\theta_1 + \theta_2)$. We are ready to construct the triangle of vectors. To start, we add the two weight forces that are both directed down (-z-axis) and make a composite vector of magnitude $Mg + M_L g$. and plot it along the negative z-axis of our reference frame. Where this vector ends, we bring another reference frame and plot F_T at θ'_T . Where the vector F_T ends, we bring another reference frame and complete the triangle by joining the tip of F_T with the origin. This vector is our third and last vector F_C . Applying cosine law, we get:

$$F_C = \sqrt{(Mg + M_L g)^2 + F_T^2 - 2(Mg + M_L g)(F_T)(\cos\theta'_T)} \qquad ....(9)$$

As shown in Fig (3c), $\theta'_T = 180 - (\theta_1 + \theta_2)$, we get:

$$F_C = \sqrt{(Mg + M_L g)^2 + F_T^2 - 2(Mg + M_L g)(F_T)\cos\{180 - (\theta_1 + \theta_2)\}}$$

Now $\cos(180 - \theta) = -\cos\theta$. We get:

$$F_C = \sqrt{(Mg + M_L g)^2 + F_T^2 + 2(Mg + M_L g)(F_T)\cos(\theta_1 + \theta_2)} \qquad ....(10)$$

In a problem in a textbook, angles θ_1 and θ_2 will be given as part of data. In a lab experiment, however, one will have to measure these angles on the table for the actual experimental set-up. A large protractor can be used for the purpose, as shown in Fig (3). Once these angles are known, the magnitude of vector F_C stands determined. As for the angle, θ_C, we shall use sine law to find the angle θ'. The angle of F_C with respect to the x-axis, θ_{CX}, is found by subtracting θ' from $90°$. This will complete the solution.

The sine law:

$$\frac{\sin\theta'}{F_T} = \frac{\sin(\theta_1 + \theta_2)}{F_C} \qquad and \qquad \theta_{CX} = 90 - \theta' \qquad ....(11)$$

where $\sin\{180 - (\theta_1 + \theta_2)\}$ has been replaced by $\sin(\theta_1 + \theta_2)$.

The Objectives of the Experiment

To verify the conditions of equilibrium for the given crane at different angles of lifting the load: (a) general case, with load (b) special case, without load.

Setting up

(a) The General Case

The crane carrying a load is shown in Fig (4): The arrangement shown in Fig (4a) is an exact parallel of the arrangement shown in Fig (3a).The same four forces act on our boom in exactly the same manner., we can, therefore, borrow Eqn: (8) for F_T :

$$F_T = \left(\frac{Mg}{2} + M_L g\right)\left(\frac{\sin\theta_1}{\sin\theta_2}\right) \qquad ....(12)$$

Measuring Angles

To proceed further in a lab, one needs to actually, physically measure angles θ_1 and θ_2. These measurements are estimates with significant inaccuracies. The process of measurement itself is cumbersome and inconvenient due to the physicality of the set-up. These errors propagate and contaminate the experimentally found values of F_T, F_C and θ_C. The level of errors is likely to exceed any decent standard of precision.

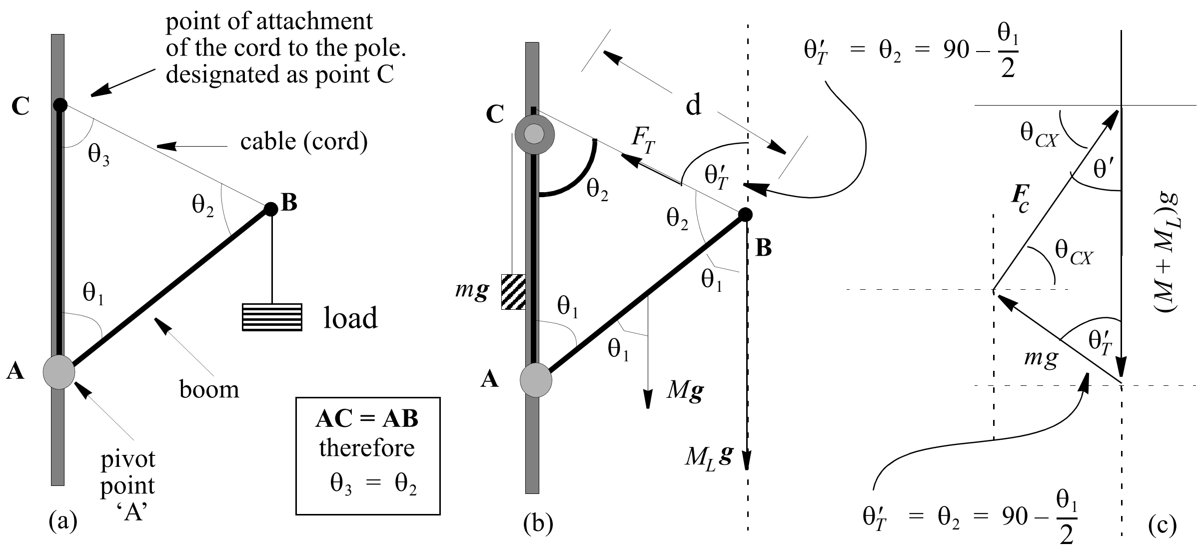

Fig (4) The Crane With Load: Introducing Elegance

Not Measuring Angles

An ambitious scheme will be to *not to* measure angles at all, by eliminating them altogether. To this end, we introduce the following elegant idea: let the boom, the cord and the relevant part of the pole, together make an isosceles triangle (ABC), with the cord being the unequal side ($\overrightarrow{BC}$). Let θ_1 be the apex angle (angle at the pivot point A) and let the two base angles each be called θ_2. The triangle is shown in Fig (4a) where:

$\overrightarrow{AB} = boom$ $\quad$ $\overrightarrow{BC} = cord$ $\quad$ $\overrightarrow{AC} = segment\ of\ pole$ $\quad$ $\overrightarrow{AB} = \overrightarrow{AC}$

$\angle BAC = \theta_1$ $\quad$ $\angle ABC = \angle ACB = \theta_2$

We can express θ_2 in terms of θ_1

$$\theta_2 = \frac{180 - \theta_1}{2} = 90 - \frac{\theta_1}{2} \qquad \text{..........(13)}$$

Next, in order to facilitate the measurement of the tension force F_T, we make use of a suspended mass used in conjunction with an unneighborly cord-pulley system. The pulley is placed in such a way that the top of the pulley coincides exactly with point C as shown in Fig (4b). Let the total suspended mass that holds the pulley in equilibrium for a specific angle of tilt θ_1, be called m. This is also shown in Fig (4b). Then

$$F_T = mg$$

Eqn (12) then changes to:

$$mg = \left(\frac{Mg}{2} + M_L g\right)\left(\frac{\sin\theta_1}{\sin\theta_2}\right)$$

Cancelling out g from the two sides and multiplying throughout by 2, we get:

$$2m = (M + 2M_L)\left(\frac{\sin\theta_1}{\sin\theta_2}\right) \qquad \text{..........(14)}$$

Eliminating One Angle:

We use two familiar trigonometric identities; one for $\sin\theta_1$ and one for $\sin\theta_2$:

$$\sin\theta_1 = 2\sin(\theta_1/2)\cos(\theta_1/2) \text{ ....}\textit{first trigonometric identity}$$

and

$$\sin\theta_2 = \sin\left(\frac{180-\theta_1}{2}\right) = \sin\left(90-\frac{\theta_1}{2}\right) = \cos\left(\frac{\theta_1}{2}\right) \text{ ....}\textit{second trigonometric identity}$$

We get, for $(\sin\theta_1/\sin\theta_2)$:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{2\sin(\theta_1/2)\cos(\theta_1/2)}{\cos(\theta_1/2)} = 2\sin(\theta_1/2)$$

Eqn (14) now reduces to:

$$2m = (M+2M_L)\big(2\sin(\theta_1/2)\big)$$

Cancelling out *2* from the two sides, we get:

$$m = (M+2M_L)\big(\sin(\theta_1/2)\big) \qquad \text{..........}(15)$$

Eliminating the Other One Too:

Let d be the geometrical length of the cord from the top of the boom to the point of its attachment to the pole. This is side $\overrightarrow{BC}$ of the isosceles triangle (Fig 4a). One can find its value by using the cosine law.:

$$(\overrightarrow{BC})^2 = (\overrightarrow{AB})^2 + (\overrightarrow{AC})^2 - 2(\overrightarrow{AB})(\overrightarrow{AC})(\cos\theta_1)$$

Letting the length $(\overrightarrow{BC}) = d$, and lengths $(\overrightarrow{AB}) = (\overrightarrow{AC}) = l$ we get:

$$d^2 = 2l^2 - 2l^2\cos\theta_1$$

$$d^2 = 2l^2(1-\cos\theta_1) \qquad \text{..........}(16)$$

From trigonometry

$$\cos\theta = 1-2\sin^2(\theta/2) \text{ ....}\textit{third trigonometric identity}$$

therefore

$$\cos\theta_1 = 1-2\sin^2(\theta_1/2) \qquad \text{..........}(17)$$

Insert this value of $\cos\theta_1$ into Eqn (16) to get:

$$d^2 = (2l^2)(1-1+2\sin^2(\theta_1/2))$$
$$= (2l^2)(2\sin^2(\theta_1/2))$$
$$= 4l^2\sin^2(\theta_1/2)$$

Taking radical on both sides, we get:

$$d = 2l\sin(\theta_1/2)$$

If a boom of length 0.5 m be used (and they really do exist), then

$$2l = 1$$

We get

$$d = \sin(\theta_1/2)$$

Switching sides,

$$\sin(\theta_1/2) = d \qquad \text{..........}(18)$$

Here d should be treated as a dimensionless number as it represents the sine of an angle. But the irony of the fate is, that it will always be measured in meters or centimeters. Writing d for $\sin(\theta_1/2)$ in Eqn (15) we get:

$$m = (M + 2M_L)d \qquad \text{..........(19)}$$

We have successfully completed the project of eliminating all angles. Now there will never be an angle to be measured!

Equation (19) matches the equation of a straight line without a y-axis intercept, in the form ($y = mx$), and could very well have been our final equation. According to the protocol, d would have been the independent variable but it turns out that its value is not unique (on account of dissipative forces that are, sadly, present). Hence we cannot *choose* its values. We shall, therefore switch sides and rewrite the equation as:

$$d = \left(\frac{1}{M + 2M_L}\right)(m) \qquad \text{..........(20)}$$

Equation (20) is the equation we shall use for our experiment. It is unbelievably simple and is totally independent of angles of all species.

(b) The Special Case

It may be viewed in some quarters that the mass of the boom may not significantly affect the analysis and as such could (or should) have been ignored. For one thing the mass is small and for another, the moment arm is one-half of that for the other two forces.

To attend to these concerns, we shall study the equilibrium of the crane without the load. Performing the experiment without the load M_L, will be quite easy. We shall just take off M_L and proceed normally for determining d for different tilt angles. Mathematical adaptation to this situation is equally easy. Just set $M_L = 0$ in Eqn (21). We get:

$$d = \left(\frac{1}{M}\right)(m) \qquad \text{..........(21)}$$

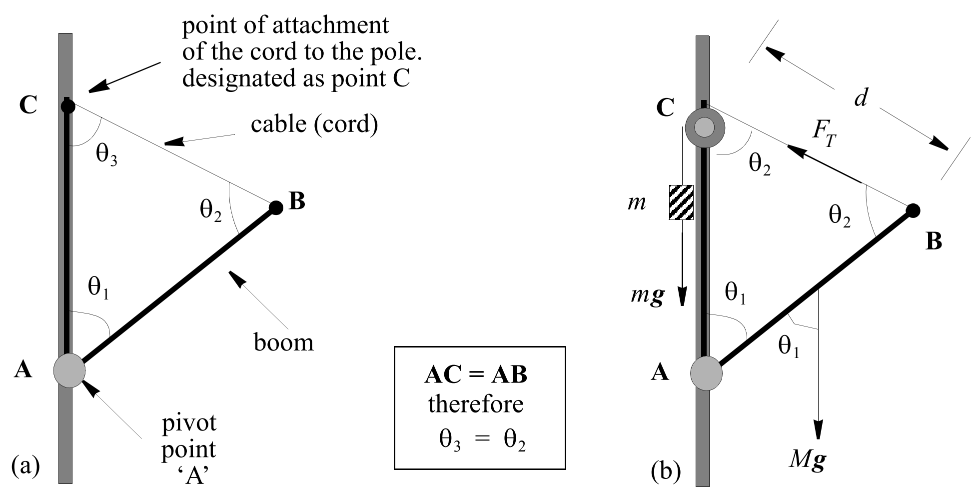

Fig (5) The Crane without the Load

The Contact Force

In all of the above discussion and analysis we neglected F_C . It is an integral part of the crane and its equilibrium. To determine F_C , we shall now make a triangle of vectors, as described earlier. Such a triangle is shown in Fig (4c). As this triangle is identical to the one given in Fig (2c), we may import the corresponding equation, Eqn (9):

$$F_C = \sqrt{(Mg + M_L g)^2 + F_T^2 - 2(Mg + M_L g)(F_T)(\cos\theta'_T)}\,. \qquad \text{..............Eqn (9)}$$

Replacing F_T by mg and pulling out g^2 from all the terms and taking its radical, we get:

$$F_C = \left[\sqrt{(M + M_L)^2 + m^2 - 2(M + M_L)(m)(\cos\theta'_T)}\right] g$$

For our crane, as shown in Fig (5c), $\theta'_T = 90 - (\theta_1/2)$. and $\cos(90 - (\theta_1/2)) = \sin(\theta_1/2)$.
This leads to: $\cos\theta'_T = \sin(\theta_1/2)$ Inserting this value in Eqn. (9), we get:

$$F_C = \left[\sqrt{(M + M_L)^2 + m^2 - 2(M + M_L)(m)\sin(\theta_1/2)}\right] g \qquad \text{..........(22)}$$

Now, from Eqn (18) $\sin(\theta_1/2) = d$. Eqn (22) can be written as:

$$F_C = \left[\sqrt{(M + M_L)^2 + m^2 - 2(M + M_L)(m)(d)}\right] g \qquad \text{..........(23)}$$

This equation allows one to calculate the magnitude of the contact force F_C .

Additionally.....

Eqn (20) allows us to express d as:

$$d = \left(\frac{m}{M + 2M_L}\right).$$

where-in all parameters are determined using a digital balance. This value of d is, therefore, far more trustworthy than the one measured between the top of the pulley and the top of the boom, using a meter stick. Because of the presence of dissipative forces, two values of d are measured and an average is taken. Hence, using our sixth sense, we quickly replace d in Eqn (23) by its value from Eqn (20). We get:

$$F_C = \left[\sqrt{(M + M_L)^2 + m^2 - 2(M + M_L)(m)\left(\frac{m}{M + 2M_L}\right)}\right] g$$

Squaring both sides,

$$F_C^2 = \left[(M + M_L)^2 + m^2 - 2(M + M_L)\left(\frac{m^2}{M + 2M_L}\right)\right] g^2$$

$$F_C^2 = \left[(M + M_L)^2 + (m^2)\left(1 - \frac{2(M + M_L)}{M + 2M_L}\right)\right] g^2 = \left[(M + M_L)^2 + \left(\frac{-M}{M + 2M_L}\right)(m^2)\right] g^2$$

$$F_C^2 = [(M + M_L)g]^2 - \left(\frac{M}{M + 2M_L}\right)(mg)^2 \qquad \text{..........(24)}$$

Rearranging,

$$[(M + M_L)g]^2 = \left[\left(\sqrt{\frac{M}{M + 2M_L}}\right)(mg)\right]^2 + F_C^2 \qquad \text{..........(25)}$$

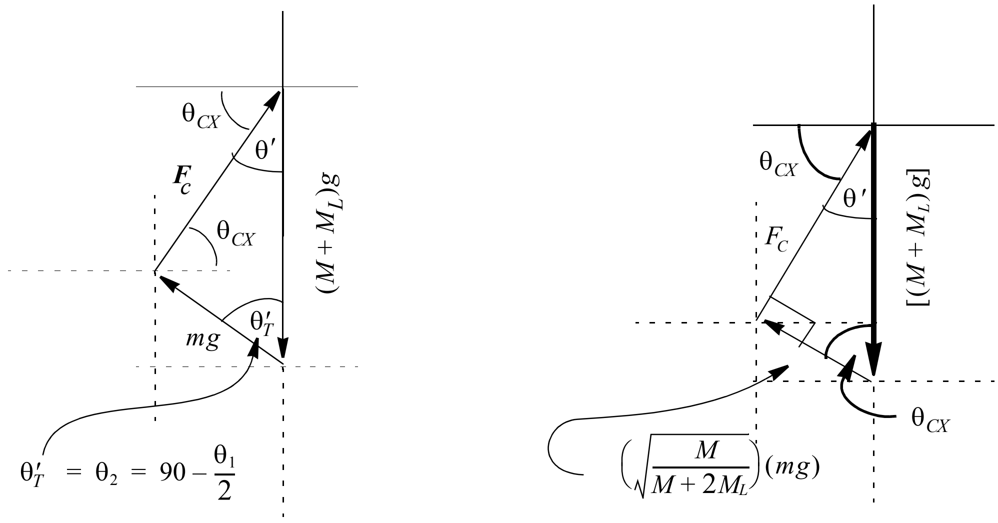

Fig (6) Converting the Triangle to a Right Angled Triangle

Eqn (25) represents a right angled triangle with the side $(M + M_L)g$ as the hypotenuse! The right angled triangle is shown in Fig (6b). The weight forces and the magnitude of the contact force remain untouched. The tension force shrinks by a constant factor

$$\sqrt{\frac{M}{M+2M_L}} = \sqrt{\frac{1}{1+2(M_L/M)}}$$

The correct direction of the angle that F_C makes with the x-axis, will still be given by the original triangle (reproduced in Fig (4c) and can be found using sine law. From Fig (4c):

$$\frac{\sin\theta'}{mg} = \frac{\sin\theta'_T}{F_C}$$

As $\theta'_T = 90 - \frac{\theta_1}{2}$ and $\sin\theta'_T = \sin\left(90 - \frac{\theta_1}{2}\right) = \cos\left(\frac{\theta_1}{2}\right) = \sqrt{1 - \sin^2\left(\frac{\theta_1}{2}\right)} = \sqrt{1-d^2}$, we get

$$\frac{\sin\theta'}{mg} = \frac{\sqrt{1-d^2}}{F_C} \qquad \text{..........(26)}$$

Eqn (26) yields θ'. The angle of F_C with respect to the x-axis, θ_{CX} (first quadrant) is then found as:

$$\theta_{CX} = 90 - \theta' \qquad \text{..........(27)}$$

This completes our determination of the contact force.

Apparatus Required

(i) A 50 cm long boom

(2) Boom holder, rod for boom holder and a table-clamp for the rod

(3) A lab jack, with a50 cm rod, pulley mounted on a rod, plumb line

(4) Cord, 2 meter long, half-meter long meter stick, one-meter long meter stick

(5) A set of masses, mass holder, digital balance, a 100 g mass for *load*.

Procedure

(a) Crane With Load (The General Case)

(1) Find the mass of the boom using the digital balance. Call it M.

(2) Find the exact mass of the 100 g mass using the digital balance. This is M_L. Suspend it at the top end of the boom, using a loop of cord.

(3) Set up the apparatus making sure that the pulley's top is *exactly* half-meter above the point of attachment of the boom (to the vertical stand) and that it is *directly above* the base of the boom. Use a plumb line if needed. Look at Fig (5b) very carefully.

(4) Select masses 20 g, 30 g,...... 130 g.

(5) Find the total suspended mass (mass selected *plus* the mass of the mass holder) using the digital balance. This is m.

(6) For each m, adjust the tilt of the boom such that it is just about balanced against being pulled upward (toward the pole). Measure carefully the length of cord (shown as $\overrightarrow{BC}$ in Fig 5b), correct to nearest millimeter. Record as d_1. For the same mass, now adjust the tilt of the boom such that it is just about balanced against being pulled downward (away from the pole). Measure the length of cord again and call it d_2. The difference of d_1 and d_2 will be small for smaller masses but will increase to many centimeters for larger masses.

Please note that larger length of cord will be needed for larger m values. This should be no problem as sufficient length of cord is available.

(7) Repeat for the remaining 11 masses.

(b) Crane Without Load (The Special Case)

(8) Remove the load M_L from the boom. Check for the set-up to be correct.

(9) Select masses: 5 g, 10 g,.....60 g on the mass holder for the 12 trials.

(10) For each mass repeat steps (5) through (7) above.

(11) The experiment ends. Remove boom from support and arrange apparatus neatly on the table

Calculations And Graphs

(a) Crane With Load (The General Case)

1) For all 12 trials, find average of d_1 and d_2 call it d_{av}. Convert to meters.
2) Convert m to kg. Plot d_{av} on y-axis and m on x-axis. Use a computer and let it fit a straight line to the points and print out the straight line equation with r^2. The equation should have ***five** significant* decimal places.
3) Take the reciprocal of the slope. It is expected to be $(M + 2M_L)$ in kg. To find the mass of the load M_L, wait till you have done the second part of the experiment.

(b) Crane Without Load (The Special Case)

4) Repeat steps (1) and (2) above for the data for this part of the experiment.
5) Take the reciprocal of the slope. It is expected to be M in kg. Compare with its expected value, found in Step (1) of procedure and find percent error.

   Use this value of M to find the mass of the load M_L, by going back to Step (3) above. Compare with its expected value. Find percent error.
6) Complete report with ***Result.s***

Additional Investigations

1) Dependence of F_C on F_T. (5 additional points)

Calculate F_C using Eqn (24), using the computer. Also calculate $F_T = mg$ on the computer. Plot F_C (y-axis), against F_T. Fit a second order polynomial The equation printed out by the computer should have 5 significant decimal places.

The intercept should match $(M + M_L)g$. Explain why, and find percent error.

2) Dependence of F_C on $\angle\theta_1$ (4 additional points)

Calculate $\angle\theta_1$ from d_{av}, using Eqn (18). Plot F_C (y-axis), against $\angle\theta_1$. Fit a second order polynomial The equation printed out by the computer should have 5 significant decimal places.

The intercept should match $(M + M_L)g$. Explain why, and find percent error.

3) Comparing $d_{av} = \frac{d_1 + d_2}{2}$ with $d_{av} = \frac{m}{M + 2M_L}$ (3 additional points)

Calculate $d_{av} = \frac{m}{M + 2M_L}$ and then plot it on x-axis with $d_{av} = \frac{d_1 + d_2}{2}$ on y-axis. Fit a straight line. The computer equation should have 3 significant decimal places.

The slope of the straight line should be unity. Departure from unity, shows your lack of expertise to locate equilibrium positions correctly. Find your percent inefficiency.

4) **Dependence of** θ_{CX} **on** $\angle\theta_1$ **(4 additional points)**

Calculate θ_{CX} using Eqns (26) and (27), and plot on y-axis, against $\angle\theta_1$. Comment on your results.

Conclusions and Discussions

Write your conclusions from the experiment and discuss them.

What Did You Learn in this Experiment?.

A hearty and thoughtful account of what you learned in this experiment by way of the principle and the techniques of experimentation, should be given

Data & Data Tables

Name.................................... **Date**..............................

Instructor.............................. **Lab Section**....................

Partner.......................... **Table #**..........................

Mass of the Boom (as found using the digital balance): $g =$ kg

Mass of the load M_L (as found using the digital balance): $g =$ kg

Total suspended mass (mass selected *plus* the mass of the mass holder) should be determined using the digital balance.

Table 1: Data for the Study of the Equilibrium of a Crane With Load

Trial #	Selected Masses (g)	Total mass m (g)	Total mass m (kg)	d_1 (cm)	d_2 (cm)	$d_{av} = \frac{d_1 + d_2}{2}$ (cm)	d_{av} (m)
1	20						
2	30						
3	40						
4	50						
5	60						
6	70						
7	80						
8	90						
9	100						
10	110						
11	120						
12	130						

Additional information or data (if any):

Total suspended mass (mass selected *plus* the mass of the mass holder) should be determined using the digital balance.

Table 2: Data for the Study of the Equilibrium of a Crane Without Load

Trial #	Selected Masses (g)	Total mass m (g)	Total mass m (kg)	d_1 (cm)	d_2 (cm)	$d_{av} = \frac{d_1 + d_2}{2}$ (cm)	d_{av} (m)
1	5						
2	10						
3	15						
4	20						
5	25						
6	30						
7	35						
8	40						
9	45						
10	50						
11	55						
12	60						

Additional information or data (if any):