Experiment # 17

Oscillations of a Rigid Body
The Physical Pendulum

Principles

Rotations of a Rigid Body

Rotational motion of a rigid body is different from its translational motion (the linear motion). In case of translational motion, there is one center-of-mass which follows a prescribed trajectory of the object's motion. The object has one inertia, called the *linear inertia.* In case of rotational motion, however, the inertia depends on the position of the axis of rotation. As the position of the axis of rotation changes, the distribution of mass of the object around the axis also changes. It is this distribution of mass that leads to *rotational inertia*. As there can be an umpteen number of axes of rotation, we conclude that a rotating object can have infinitely many rotational inertias.

Complex as the situation may appear to be, we have enough rules to determine these inertias and as such are confident of being able to deal with any type of rotation of a rigid body.

A rigid body may rotate about an axis lying inside its body or outside. If the axis of rotation lies outside the body of the object, the object may be treated as a discrete particle of mass m. The rotational inertia of the object, in this case, is given by:

$$I = mr^2 \qquad \text{..........(1)}$$

where r is the distance of the center-of-mass of the object from the axis of rotation. If however, the rigid body is rotating about an axis that lies within the body of this body, then its rotational inertia is given by the formula:

$$I = I_{cm} + Mh^2 \qquad \text{..........(2)}$$

where M is the mass of the rigid body, I_{cm} is its rotational inertia about an (unused) axis passing through the center-of-mass. h is the distance of the actual (used) axis of rotation of the rigid body from its center-of-mass. It is assumed that the two axes are parallel to one another. Eqn (2) is known as the Law of Parallel Axes.

Oscillations of a Rigid Body

A rigid body may rotate round and round in a horizontal circle of radius R, if the axis is vertical. If the axis is horizontal then it will go round and round in a vertical circle of radius R; or, in case enough excitation is not available, it may just move back and forth in the vertical mode, about a mean position.

The back and forth (or, the to-and-fro) motion of the rigid body about a mean position in the vertical mode is called *oscillation*. Oscillations may be of several different types. If we plot time on the x-axis and displacement on the y-axis, we shall get what we call a *waveform*. It is a visual display of the oscillatory motion. The waveform may be square, rectangular, triangular or of any other regular or irregular shape or form. As long as the pattern repeats itself, it will be called an oscillation.

Of particular interest is the oscillation where the waveform is sinusoidal. It is known as *Simple Harmonic Oscillation*. Familiar examples of this type of motion are the oscillations of pendulums that we find in *grand father* clocks of different types. Oscillations of springs, in the vertical mode (when a mass is attached to it) are also simple harmonic. The vibrations of the surface of a drum are two-dimensional simple harmonic oscillations. Objects that execute simple harmonic oscillations, are called Simple Harmonic Oscillators.

The mathematical equations representing the sinusoidal oscillations of a simple harmonic oscillator are:

$$x(t) = a\sin(\omega t) + b\cos(\omega t) \qquad \text{..........(3)}$$

$$x(t) = A\cos(\omega t + \phi) \qquad \text{..........(4)}$$

where ϕ is the phase angle and $\omega = 2\pi\nu = (2\pi)/T$

Analyzing Oscillations

Consider a rigid body of irregular shape and size *and* of irregular mass distribution. One such body is shown in Fig (1). Let it be rotated about an horizontal axis passing through O in the (x,z) plane, where O is *not* the object's center-of-mass.

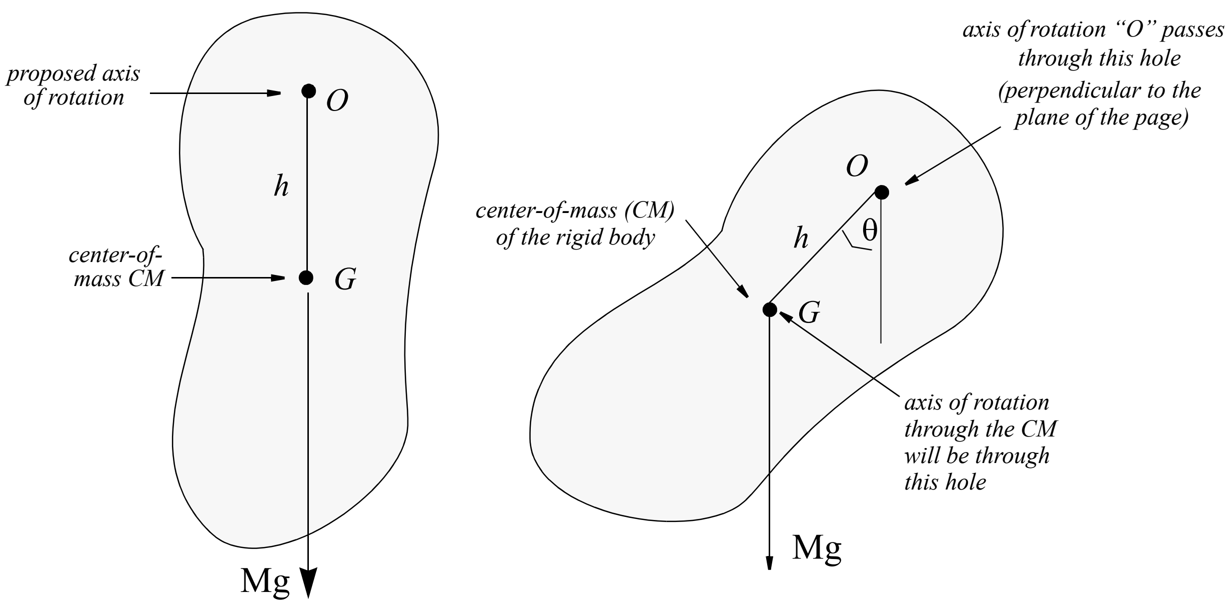

Fig (1) Oscillations of a Rigid Body

Let the center-of-mass of the body be at G. The weight force of the body Mg acts at this point. To begin with, the body will be in a position where G lies below O. This is shown on the left side in Fig (1). If we now displace the body sideways by a small angle θ and hold it there, then it will be subjected to a restoring torque exerted upon it by the weight force Mg.

$$\tau_{Mg} = (Mg)(h)(\sin\theta) = Mgh\sin\theta \qquad \text{..........(5)}$$

where h is the distance of O from G. This being the only torque, we write:

$$\tau_{net} = -\tau_{Mg} \quad \text{..........(6)}$$

where the negative sign has been assigned because it is a *restoring* torque and acts in a direction opposite to that of the torque that displaced it in the first place. By the second law of motion for rotational motion, we have:

$$\tau_{net} = I\alpha \quad \text{..........(7)}$$

From Eqns (5), (6) & (7) we get:

$$Mgh\sin\theta = -I\alpha$$

Let the angle θ (by which the rigid body is being displaced) be small enough such that $\sin\theta = \theta$, where $\angle\theta$ is in units of rads/sec. Then we get:

$$Mgh\theta = -I\alpha \quad \text{..........(8)}$$

At this time we recall that

$$\alpha = \frac{d^2\theta}{dt^2}$$

Inserting this value of α in Eqn (8) we get:

$$Mgh\theta = -I\frac{d^2\theta}{dt^2}$$

Or

$$\frac{d^2\theta}{dt^2} + \frac{Mgh\theta}{I} = 0 \quad \text{..........(9)}$$

In Eqn (9) M, g, h, and I are all constants. If we let:

$$\frac{Mgh}{I} = \kappa^2$$

then we can write Eqn (9) as:

$$\frac{d^2\theta}{dt^2} + \kappa^2\theta = 0 \quad \text{..........(10)}$$

It is very interesting to consider the dimensions of κ^2 which has replaced Mgh/I:

Unit of M is kg, that of g is m/s^2, that of h is m, and that of I is kgm^2

Combining we get:

$$(kg)(m/s^2)(m)(1/kgm^2) = 1/s^2 = (1/s)^2$$

But $(1/s)$ is the unit of ω. We find that our constant κ is in fact just ω. We get:

$$\frac{Mgh}{I} = \omega^2 \quad \text{..........(11)}$$

We were sort of expecting this. This is because Eqn (10) is in the form of equation of motion, and as such, it was expected to have a term representing the velocity of the object (angular velocity ω, in this case). But it is interesting to see how it emerged just from a consideration of units!

Our equation of motion has now taken the proper form. It is written as:

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad \text{..........(12)}$$

We need not, however, solve this equation as we are only interested in the time period T, of the *small* oscillations of the simple harmonic motion of this rigid body. From Eqn:(11):

$$\omega = \sqrt{\frac{mgh}{I}}$$

Converting circular frequency ω to linear frequency f, we get:

$\omega = 2\pi f$ and $f = 1/T$ therefore $\omega = (2\pi)/T$

Hence

$$\frac{2\pi}{T} = \sqrt{\frac{mgh}{I}}$$

Or

$$T = (2\pi)\sqrt{\frac{I}{Mgh}} = (2\pi)\sqrt{\frac{I/(Mh)}{g}} \qquad \text{..........(13)}$$

This is the time period T of the small oscillations of our rigid body.

Next we proceed to determine the rotational inertia I of the rigid body, for rotation about the axis through O. Let the rotational inertia of the body through its center-of-mass be called I_{cm}. This will be the rotational inertia about an axis through G because G is the center-of-mass of this rigid body. As the two axes of rotation (one through G and the other through O) are parallel axes, we shall use the theorem of parallel axes to find the rotational inertia of the body about an axis through O. Since I is the rotational inertia through O, the theorem of parallel axes leads us to:

$$I = I_{cm} + Mh^2$$

We plug in this value of I in Eqn (13), to get:

$$T = (2\pi)\sqrt{\frac{(I_{cm} + Mh^2)/(Mh)}{g}} \qquad \text{..........(14)}$$

Let κ be the radius of gyration of the rigid body, then its I_{cm} is given by:

$$I_{cm} = M\kappa^2$$

Inserting this value of I_{cm} in Eqn (14) we get:

$$T = (2\pi)\sqrt{\frac{(M\kappa^2 + Mh^2)/(Mh)}{g}}$$

Cancelling out M we get:

$$T = (2\pi)\sqrt{\frac{(\kappa^2 + h^2)/h}{g}} \qquad \text{..........(15)}$$

This is another very exciting result! We find that the expression for the time period of the rigid body (which has turned out to be independent of mass), has exactly the same form as the expression for the time period of a simple pendulum:

$$T = (2\pi)\sqrt{\frac{l}{g}} \qquad \text{..........(16)}$$

We note that the two equations (#s 15 & 16) will match exactly if we were to set:

$$\frac{(\kappa^2 + h^2)}{h} = l$$

where l is the length of the (massless) cord of the *simple* pendulum. As the quantity $(\kappa^2 + h^2)/h$ does have the unit of length, we may call it *the equivalent length* of the rigid body. With this *length* the rigid body will behave as a *simple* pendulum. Using our scientific ingenuity, we have rendered this rigid body into a simple pendulum!

To find this *equivalent* length l of the rigid body, let us write:

$$l = \frac{(\kappa^2 + h^2)}{h}$$

Rearranging, we get a quadratic equation in h, where h is the distance of point O from the center-of-mass G of the rigid body:

$$h^2 - lh + \kappa^2 = 0 \qquad \text{..........(17)}$$

We shall get two values of h, for which the time period T (vide Eqn # 15) will be the same. We will not attempt to solve the equation. Instead we shall make use of the properties of quadratic equations. Recall the quadratic equation in x:

$$ax^2 + bx + c = 0$$

It gives us two solutions for x. According to the properties of the quadratic equation, the sum of the solutions and the product of the solutions are respectively given by:

$$x_1 + x_2 = -b/a \qquad x_1x_2 = c/a$$

Applying to our quadratic equation, we find that (since $a = 1$):

$$h_1 + h_2 = l \qquad \text{..........(18)}$$

$$h_1h_2 = \kappa^2 \qquad \text{..........(19)}$$

The equivalent length l of our rigid body is, thus, found as the sum of two lengths, h_1 and h_2, each being the distance of an axis of rotation of the rigid body along the line OG, as shown in Fig (1).

One way of finding the two values of h is to select a large number of axes of rotation along the line OG and experimentally find the time period T, for each axis. Because of the non-linearity, the graph will not be a straight line. It will be parabolic (why?).

For improved accuracy, one can *reverse* the rigid body by extending the line OG to the other side of G to some point O' such that the length $O'G$ equals length OG and repeat the experiment. This is shown in Fig (2). We shall get two sets of values of h_1 and h_2. We can then calculate a mean value for each of h_1 and h_2.

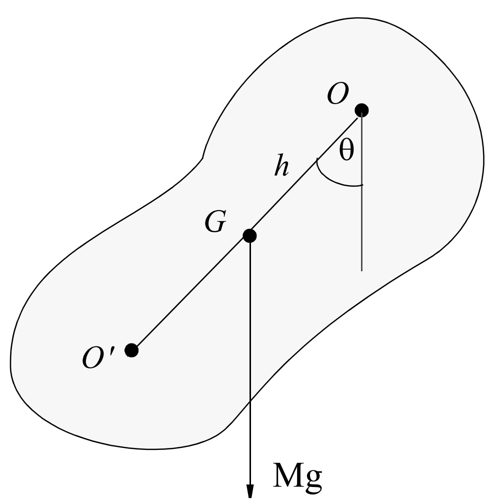

Fig (2) The Line OGO'

From Eqn (19) we get:

$$h_2 = \frac{\kappa^2}{h_1}$$

If we plug in this value of h_2 in Eqn (18) we get:

$$h_1 + \frac{\kappa^2}{h_1} = l$$

Rearranging:

$$\frac{\kappa^2 + h_1^2}{h_1} = l \qquad \text{..........(20)}$$

Similarly if we repeat this procedure for h_2, we shall find that:

$$\frac{\kappa^2 + h_2^2}{h_2} = l \qquad \text{..........(21)}$$

From Eqns (20) & (21) we learn that the time period T for the two lengths h_1 and h_2 will be the same. If we start out from a large value of h and keep decreasing it, the time period T will also decrease. It will reach a minimum value and then will begin to increase, *while still on the same side* of the center-of-mass G, along the line OG. This will lead us to getting identical time period for two different h values. These two different h values will be h_1 and h_2! The same will happen for axes on the other side of the center-of-mass, along the line $O'G$.

A typical graph is shown in Fig (3)

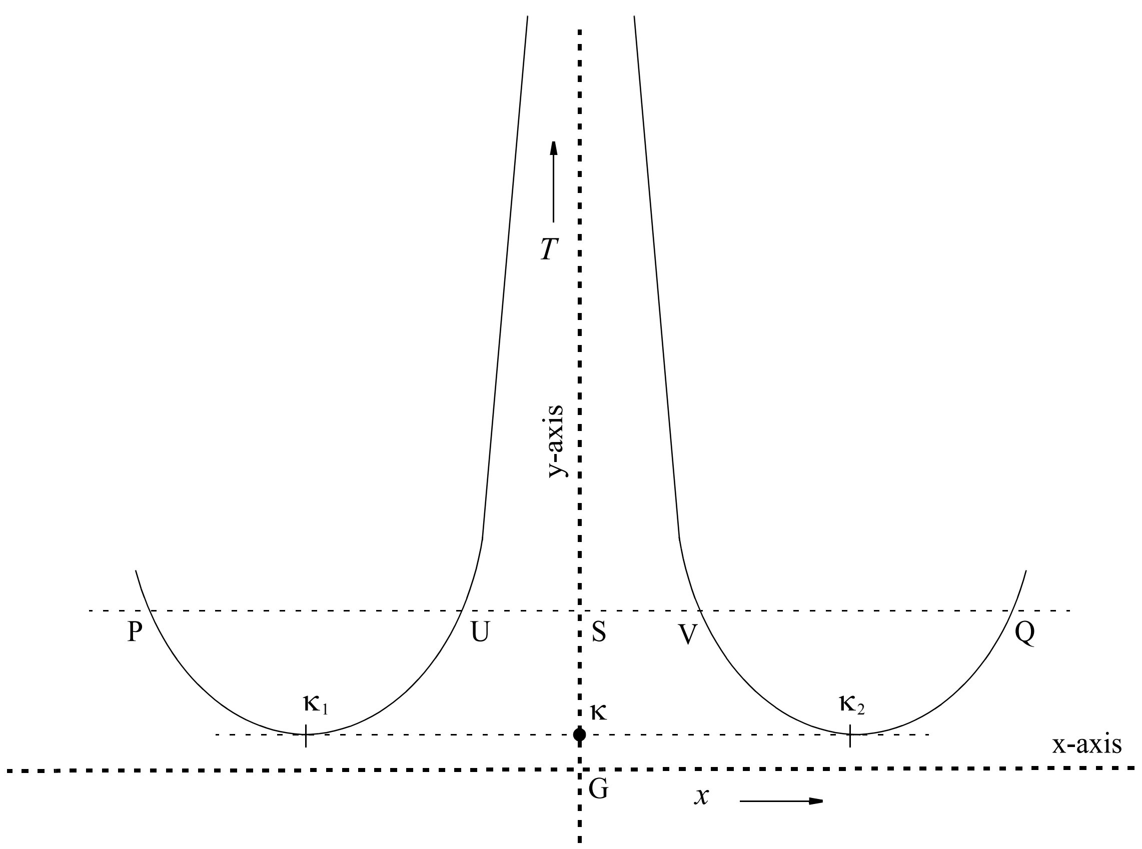

Fig (3) Plotting T against h

Objectives of the Experiment

(A) Determine the (a) equivalent simple pendulum length, and (b) the radius of gyration of the given physical pendulum (i) without the accessory mass, and (ii) with the accessory mass.

(B) Find the value of g, the acceleration due to gravity from each set of measurement; compare with its actual value and find percent errors.

Setting Up

The rigid body used for the experiment is known as *Physical Pendulum*. It consists of a long uniform bar with about 15 holes on each side of the central hole. The central hole represents the center-of-mass of the bar. Even though the bar is symmetrical with respect to the center-of-mass, the holes are not drilled at equal distances. Thus the distribution of mass along the bar is not uniform. It will not be possible to apply formulae for the inertial mass of a uniform bar, in this case. The time period for the oscillation of the bar with the axis of rotation passing through the central hole, will be infinite. This is evident from Fig (3) where the curve is seen to rise to infinity in the vicinity of the point *G*.

We shall set the pendulum into oscillations using as many axes as possible and find the corresponding time periods. After plotting the double graph, we shall select a suitable value of the time period T on the y-axis (the time axis) and draw a straight line parallel to the x-axis. This will cut the curves at 4 points. These are marked as points P, Q, U, and V. The value of the time period is to be read at point S.

Lengths SP and SU will respectively be h_1 and h_2, the two values of h for which the theory predicts identical time periods. We shall get another pair of their values from lengths SQ and SV. It will be in order to calculate averages of h_1 and h_2. Use these average values to find the equivalent length l, using Eqn (18). Once l has been determined in this *glorious* manner, we may calculate the value of g using the simple formula of the simple pendulum.

Additionally it is to be stated that the lengths $\kappa\kappa_1$ and $\kappa\kappa_2$ are expected to be equal. Each is supposed to be equal to the radius of gyration κ of the rigid body.

Apparatus Required

(1) Physical Pendulum with accessory mass and knife edge support,

(2) Electronic timer with Photogate

(3) Ruler and meter stick.

Procedure

A typical physical pendulum is shown in Fig (4), with and without the accessory mass. Arrangement to measure time period using an electronic timer has also been shown.

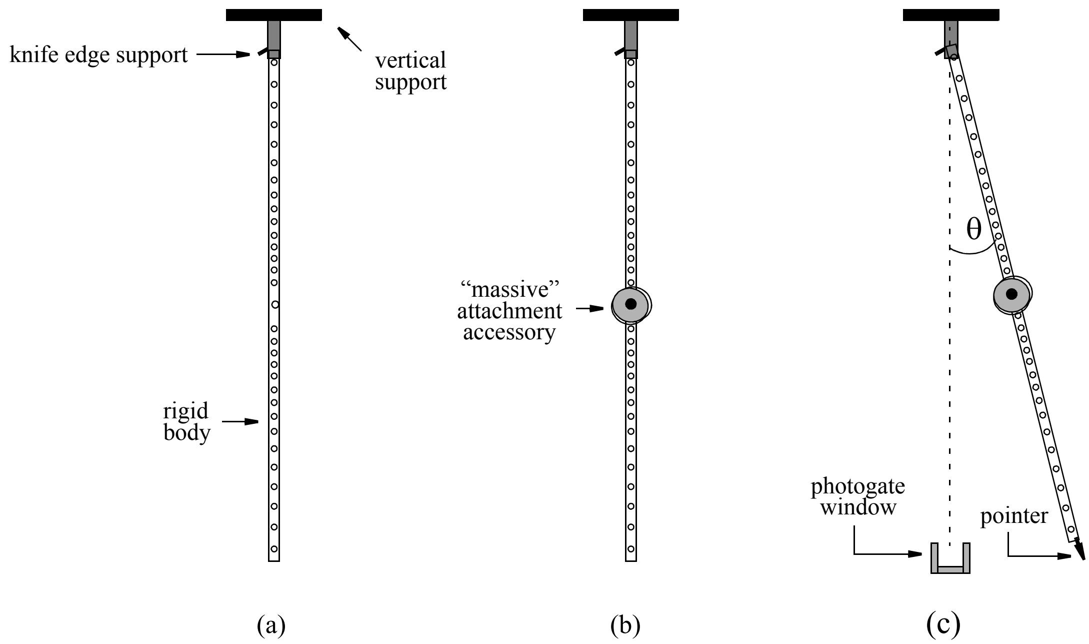

Fig (4) Physical Pendulum

(1) Take the pendulum bar off the support, remove the heavy weight screwed on, at its center and lay the bar flat on the table.

(2) Choose a *right* and a *left* side of the pendulum and make appropriate mark on each side using a marker pen.

(3) Measure lengths h from the central hole to the holes on (i) the right side, and (ii) the left side. Call these h_{1r}, h_{2r}, ... and h_{1l}, h_{2l}, ... and record in the data sheet.

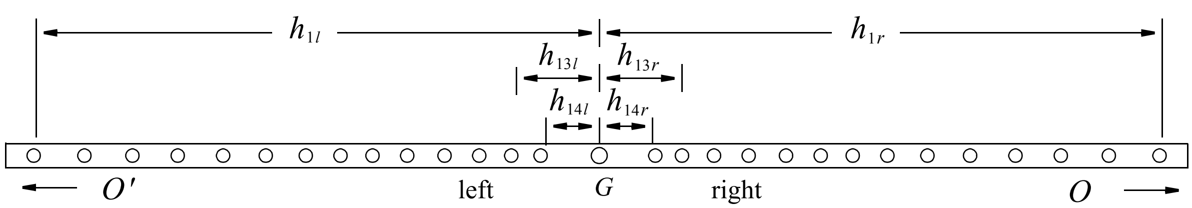

Fig (4) Measuring Lengths

Pendulum Without "Accessory Mass"

(4) Set up the pendulum with the top most hole of the *right* side in the knife-edge support. Attach the small plastic pointer at the lower end of the pendulum. Place the window of the photogate timer such that the tip of the pointer passes through the window thereby interrupting the beam.

(5) Set the electronic timer in the ***Physical Pendulum*** mode, or simply, the ***Pendulum*** mode. Set it to record 10 consecutive time periods by selecting 10 memories. The timer will display the average time period when the *enter* key is depressed. Each value of time will be measured to one tenth of a millisecond.

(6) Displace the pendulum through a small angle (less than 5°) and gently release it from rest. Make sure the bar does not wobble. If it does, stop it manually and try again.

(7) After the pendulum has completed couple of trouble-free oscillations, clear the timer.

(8) The pendulum will oscillate about its mean position and the timer will measure the necessary values of time. After the timer stops, stop the pendulum manually. Read and record the value of $(T_{1r})_{av}$.

(9) Remove the pendulum from its support and suspend it from the second hole from the top, on the *right* side and repeat steps 6, 7, and 8.

(10) Repeat steps 9 for the remaining holes on the *right* side.

(11) Repeat the above for all the holes on the *left* side of the pendulum.

Pendulum With ***Accessory Mass***

(12) Attach the two heavy disk-masses (the accessory masses) in the central hole of the pendulum, using the screw. You may need a screw driver to tighten the screw.

(13) Repeat steps (6) through (11) for the pendulum with the accessory masses.

(14) The experiment ends. Switch off the timer and unplug it. Suspend the pendulum from one of its outer-most holes.

Calculations & Graphs

(1) Enter values of h_{1r}, h_{2r}, ... as positive numbers in the data sheet of the computer. Then enter values of h_{1l}, h_{2l}, ... as negative numbers

(2) Choose h_{1r}, h_{2r}, ... (positive numbers) for the x-axis of the graph and choose the corresponding values of the average time periods $(T_{1r})_{av}$, $(T_{2r})_{av}$, ... for the y-axis. Choose *scatter* graph from the graph menu. In the curve fitting menu, choose *interpolate.* The computer will fit a smooth curved line, on the entire page.

(3) Choose h_{1l}, h_{2l}, ... (negative numbers) for the x-axis of the graph and choose the corresponding values of the average time periods $(T_{1l})_{av}$, $(T_{2l})_{av}$, ... for the y-axis. From the Graph menu, choose *overlay* graph. The computer will plot the data on the same graph by shrinking the first graph to the right half of the page. It will plot the new data on the left half of the page. In the curve fitting menu, choose *interpolate.* The computer will fit a smooth curved line, to the new data. Please note that the two curves will be found to be oppositely oriented.

(4) Choose scales for the x- and y- axes that will enable you to read lengths on the two axes with reasonable accuracy. Ask the instructor for details of scale selection. You may also need help for drawing straight lines on the graph, while still on the screen.

(5) Choose a convenient value of the time period T and draw a line parallel to the x-axis. Next draw a line parallel to the x-axis and tangential to the two curves at their bases; as shown. in Fig (3). Now print the graph and the data sheet.

(6) Mark points P, Q, U, V and S as shown in Fig (3) on the graph paper. Read lengths SP, and SQ. Find the average value. This your $(h_1)_{av}$. Carefully read lengths SU and

SV. Find the average value. This is $(h_2)_{av}$. Use these values to calculate l as per Eqn (18). Calculate g using Eqn (16). Compare with its actual value (9.80 m/s^2) and find percent error.

(6) Repeat step (5) for the second set of data for the pendulum with the accessory mass.

(7) For each graph read lengths $\kappa\kappa_1$ and $\kappa\kappa_2$, on the appropriate line. Find average of the two values and record the average length as the *radius of gyration* of the pendulum. Please note that the radius of gyration always refers to the axis of rotation being at the center-of-mass of the object.

(8) Complete the report with ***Results***.

Conclusions and Discussions

Write your conclusions from the experiment and discuss them.

What Did You Learn in this Experiment?.

A hearty and thoughtful account of what you learned in this experiment by way of the principle and the techniques of experimentation, should be given

Data & Data Tables

Name.................................... **Date**..............................
Instructor.............................. **Lab Section**....................
Partner.......................... **Table #**..........................

Table 1a Determining the Time Periods Without the Accessory Mass

Hole # n	Length h_{nr} m	Time period (sec) $(T_{nr})_{av}$	Length h_{nl} m	Time period (sec) $(T_{nl})_{av}$
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				

Table 1a Determining the Time Periods With the Accessory Mass

Hole # n	Length h_{nr} m	Time period (sec) $(T_{nr})_{av}$	Length h_{nl} m	Time period (sec) $(T_{nl})_{av}$
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				