Experiment # 15

Rotational Dynamics (1)
Rotational Inertia

Principles

Introduction

Rotational dynamics deals with the accelerated motion of a rigid body in a circle or in a part thereof. The radius of the circle r, will stay constant for every part of this study. The center of the circle serves as the axis of rotation. The prerequisite of such a motion is of course, the centripetal acceleration $\boldsymbol{a}_{cp}$. It is directed radially inward and is given by the formula:

$$\boldsymbol{a}_{cp} = v^2/r$$

Kinematics: (a) Tangential

We may study the kinematics of a rotating object in the reference frame of the rotating object itself. This amounts to riding with the rotating object and monitoring (i) the *arc distances* traversed by the rotating object along the circumference, and (ii) the corresponding time durations consumed by the object. These two measurements determine the instantaneous velocities that are tangential to the circumference. Changes in *instantaneous tangential velocities* will naturally be effected by a constant acceleration that would be tangential to the circumference too (but not instantaneous, for reasons of being constant). Unknowingly we have developed the *tangential* mode of motion, as different from other modes that we know. To use this mode effectively we *import* the four kinematic equations that deal with all aspects of motion in a given mode. The imported equations are listed in column (1) of Table (1), without of course, the bulky subscript ***tang*** that you may have expected to find glued to the *tangential velocity* and the *tangential acceleration*. We have also used letter l for *arc distances* in place of the letter d that was used for *linear displacement*.

Kinematics: (b) Angular

One can also study the kinematics of a rotating object in a reference frame located at the center of the circle of rotation. Draw a line from the center-of-mass of the (rotating) object to the axis of rotation (the center of the circle). This line will simply be the radius r of the circle of rotation. As the center-of-mass of the object moves, through an arc distance l, the said line sweeps an angle θ at the center. Elements of geometry tell us that

$$\theta = l/r \qquad \text{..........(1)}$$

As r is forever constant, we find that θ and l are directly proportional to each other. This gives us an idea. Instead of monitoring *arc distances* l, why not monitor *angular distances* θ*?* Monitoring angular distance θ and the corresponding time intervals, will lead us to *instantaneous angular velocity* instead of the *instantaneous tangential velocity.* Continuing the line of argument, we soon develop the concept of *angular acceleration* which will not be *instantaneous* for reasons of being constant. They chose the mathematical symbols ω for the (instantaneous) angular velocity and α for the (constant) angular acceleration. Having developed a fresh mode of motion, all that remains for us to do, is to *import* a fresh set of the four kinematic equation. The successfully imported equations are shown in column (2) of Table (1).

Table 1: Kinematic Formulae For Rotational Motion

	tangential mode	angular mode
1	$v = v_o + at$ ....*(2)*	$\omega = \omega_o + \alpha t$ ....*(3)*
2	$l = v_o t + (1/2)at^2$ ....*(4)*	$\theta = \omega_o t + (1/2)\alpha t^2$ ....*(5)*
3	$l = (1/2)(v + v_o)t$ ....*(6)*	$\theta = (1/2)(\omega + \omega_o)t$ ....*(7)*
4	$l = (v^2 - v_o^2)/(2a)$ ....*(8)*	$\theta = (\omega^2 - \omega_o^2)/(2\alpha)$ ....*(9)*
5	$l = r\theta \quad v = r\omega \quad a = r\alpha$ ....*(10)*	$a_{cp} = v^2/r = \omega^2 r$ ....*(11)*

Kinematics: Inter convertibility

The two modes of studying the kinematics of rotational motion are completely different from one another but a transition from one system to the other is possible and can be effected at any stage. This is based on the geometrical relationship between arc length l and the angular distance θ, vide Eqn (1) above. This leads to the conversion relations shown in the fifth row of Table (1). It should be noted that the tangential velocity v is not considered to be a vector quantity. Instead, the angular velocity ω, is designated to be the vector quantity. It is directed along the axis of rotation. Thus if the circle of rotation is in the (x,y) plane and the object rotates from +x towards +y (in a counter clockwise direction) then ω is directed along the +z axis

Dynamics

The agency that produces acceleration (tangential or angular) is called *torque* and we write τ for it. It is basically a force, applied in a special way and therefore it is not a basic entity. For this reason it has two definitions: the **structural** definition, and the **functional** definition.

Torque: The Structural Definition

According to the structural definition,

$$\tau = Fr\sin\theta \qquad \text{..........(12)}$$

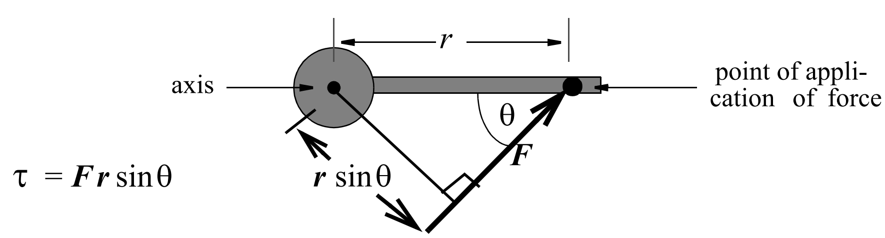


Fig (1) The Structural Definition of a Torque where F is the applied force, r is the distance of the point of application of force from the axis of rotation, and θ is the angle between F and r. This is shown in Fig (1). It can also be written as the product of F and $r\sin\theta$, where $r\sin\theta$ (see the diagram) is the *moment arm* of the torque.

Torque: The Functional Definition

The functional definition is in terms of the function of the torque, which is to produce angular acceleration α. According to this definition:

$$\tau = I\,\alpha \qquad \text{..........(13)}$$

where I is the *constant of proportionality.* It is recognized as *rotational inertia* of the rotating object.

From these two definitions (Equations 12 and 13) one can evolve another useful equation by eliminating τ:

$$Fr\sin\theta = I\alpha = I(a/r) \qquad \text{..........(14)}$$

Rotational Inertia

An object moving in a straight line has linear inertia whereas the same object rotating in a circle, will have rotational inertia. Linear inertia represents the distribution of mass around the center-of-mass of the object. Rotational inertia, on the other hand, has to do with the distribution of mass around the axis of rotation which may or may not be at its center-of-mass. Again, an object has only one value for its linear inertia but it may have infinitely many values for its rotational inertia. This is because an object can rotate around any of the infinitely many axes of rotation, which may be located inside or outside the body of the object.

We divide rotation into two categories:

(a) Orbiting Motion: Rotation of the object about an axis outside the body of the object, or rotation of the object about an *external axis.*

(b) Spinning Motion: Rotation of the object about an axis inside the body of the object, or rotation of the object about an *internal axis.*

Rotation of an Object (or System of Objects) About an External Axis

The inertia of an object in this case is given by the equation: $I = mr^2$. If we are dealing with a system of objects such that the common axis of rotation is outside the bodies of all participating objects in the system, then the inertia of this system-of-particles is given by the equation:

$$I = \Sigma m_i r_i^2 \qquad \text{..........(15)}$$

Here r_i are the distances of the centers-of-mass of the participating objects from their (common) axis of rotation.

Rotation of an Object About an Internal Axis

For the sake of simplicity, we shall consider an object rotating about an axis passing through its center-of-mass. The inertia in this case is given by the equation:

$$I = MK^2$$

where K is *radius of gyration* of the object. We define K as the distance from the center-of-mass of the object to the point where a torque *effectively* acts (or gets applied). Values of K depend on the geometrical shape and size of objects and are found using calculus. The radius of gyration K is always less than or equal to the radius of the circle of rotation R. For this reason K is expressed as a fraction of R.

If the axis does not pass through the center-of-mass of the object, but is located some distance h away from it (still lying within the body of the object), then the inertia is given by the equation:

$$I = MK^2 + Mh^2 = M(K^2 + h^2)$$

Values of K for three commonly used objects are given in Table (2). The table also lists the values of their traditional rotational inertia I.

Table 2: Rotational Inertias of Some Common Objects

Object	I in terms of K	K as fraction of R	I in terms of R
Disc or cylinder	$I = MK^2$	$K = 0.5R$	$I = (1/2)MR^2$
Sphere	$I = MK^2$	$K = 0.4R$	$I = (2/5)MR^2$
Ring	$I = MK^2$	$K = R$	$I = MR^2$

Inertia As An Additive Quantity

Rotational inertias must be calculated for each part of a system and then added scalerly. For some members of a system, the axis may be outside and for others it may be inside. A good example is that of children on a merry-go-round.

Work And Energy

A rotating system has rotational kinetic energy, KE_{rot}, and angular momentum L. The work done by a rotating system is W joules. The formulae for these are given below. These are exact parallels of corresponding formulae for linear motion. The linear force, F is replaced by the *rotational force* τ. Similarly linear inertia, m is replaced by rotational inertia I and linear displacement, d, is replaced by angular displacement, θ.

$$KE_{rot} = (1/2)I\omega^2 \qquad L = I\omega \qquad W = \tau\theta \qquad \text{..........(16)}$$

Law of Conservation of Energy

The law of conservation of energy holds for the rotating systems also. The discussion of the law will not be repeated here. Instead we would refer to the discussion given in the experiment: “Energy Conservation (1)”. We shall, however, add the “rotational kinetic energy” to the list of mechanical energies; the modified statements of the law are:

$$\Delta E = \Delta KE_{rot} + \Delta KE_{rot} + \Delta PE_g + \Delta PE_s \qquad \text{..........(17)}$$

$$\Delta E = 0 \qquad F_{nc} \text{ are absent} \qquad \text{..........(18)}$$

$$\Delta E = W_{nc} \qquad F_{nc} \text{ are present} \qquad \text{..........(19)}$$

Objective of the Experiment

To determine the rotational inertia I of the given system of discs.

Setting up

The rotational inertia of the given system of discs can be determined using the structural and the functional definitions of the torque, as given by Equations (12) & (13), above.

Determining the torque τ.

To be able to make use of the kinematic and dynamic equations, one would need to apply a ***uniform*** torque τ, on the system of discs to impart it a constant angular acceleration α. A knowledge of τ and α should enable one to determine I.

A uniform torque is best applied by using a suspended mass. A mass holder holding the suspended mass is connected to the system of discs via an unneighborly set of *cord and pulley.*

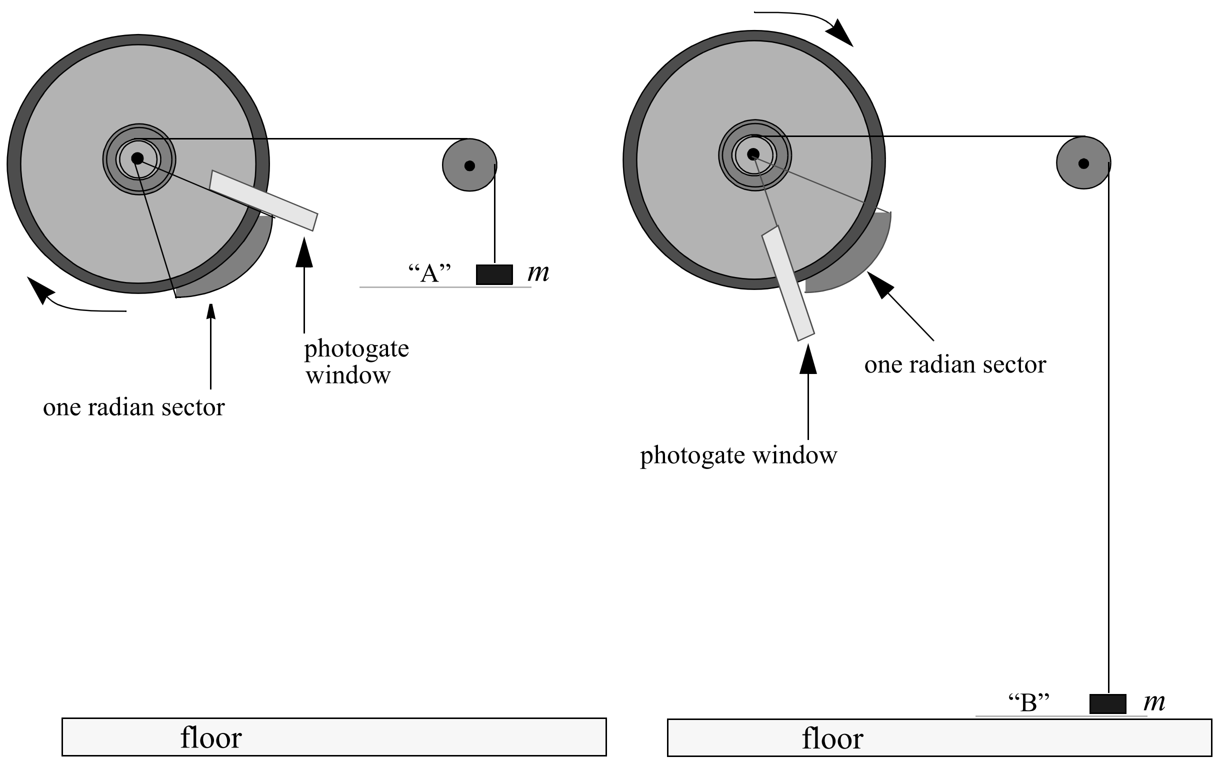


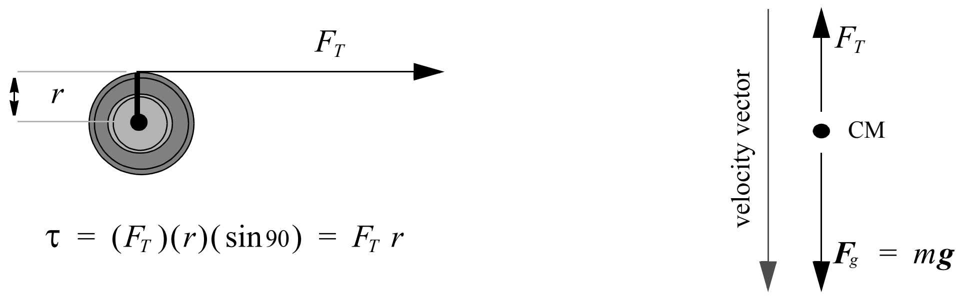
Fig (2) A Uniform Force Exerts a Uniform Torque on the System of Discs

This pulley is mounted at the end of the table. The suspended mass pulls on the system of discs and rotates it. Such an arrangement is shown in Fig (2).

The Analysis

The experimental set-up consists of two objects: (i) the system of discs, and (ii) the suspended mass. Necessary force diagrams for these appear in Fig (3).

(i) The system of discs have a set of aluminum pulleys mounted on top of the discs. These are of much smaller radii compared to those of the discs. Being aluminum pulleys, their mass is much smaller than those of the discs. The idea is to have pulleys without adding (significantly) to the inertia of the discs. The discs themselves are mounted on a spindle that is loaded with ball bearings. This arrangement reduces losses due to dissipative forces to a minimum.



(a) The aluminum pulley (attached to the system of discs)

(b) The suspended mass m

Fig (3) The Force Diagrams

In view of the above, the *only* force that acts on the system of discs is the tension force, F_T, developed in the cord by the suspended mass. This is shown in Fig (3a). The force acts tangentially and its distance from the axis of rotation is the radius of the aluminum pulley, r. The *only* torque (hence τ_{net}), therefore, is:

$$\tau_{net} = (F_T)(r)(\sin 90) = F_T r \qquad \text{..........(20)}$$

As $\tau_{net} = I\alpha$, we get

$$F_T r = I\alpha$$

Or

$$F_T = \frac{I\alpha}{r} \qquad \text{..........(21)}$$

(ii) The force diagram for the suspended mass is shown in Fig (3b). As the suspended mass moves down, the net force F_{net} acting on the suspended mass is $(mg - F_T)$. Applying Newton's Second law, we get:

$$mg - F_T = ma_{linear} \qquad \text{..........(22)}$$

The system of discs has acceleration α and the suspended mass has acceleration a_{linear}. To have a homogeneous acceleration, we first convert a_{tang} of a point on the rim of the aluminum pulley (radius r) to α. This gives us:

$$a_{tang} = r\alpha \qquad \text{..........(23)}$$

Next we use the principle:

If the cord winds or unwinds without slipping from the pulley, then the tangential acceleration of a point on the rim of the pulley is equal to the linear acceleration of the suspended mass.

$$a_{linear} = a_{tang} \qquad \text{..........(24)}$$

From Eqns (23) and (24), we get

$$a_{linear} = r\alpha \qquad \text{..........(25)}$$

Inserting this value of a_{linear} in Eqn (22), we get:

$$mg - F_T = mr\alpha \qquad \text{..........(26)}$$

Now insert the value of F_T from Eqn (21) into Eqn (26), to get:

$$mg - \frac{I\alpha}{r} = mr\alpha$$

Multiply throughout by r and take the $I\alpha$ term to the right hand side:

$$mgr = (I + mr^2)\alpha \qquad ....(27)$$

This is as far as the analysis will take us. Eqn (27) does not match the equation of a straight line because the variable m, could not be isolated so as to remain only on one side of the equation. It occurs on both sides of the equation and there is no way the situation can be resolved mathematically.

Thinking desperately for a non-mathematical prescription, we consider the relative magnitudes of the two terms: I and mr^2. We find that, in fact mr^2 is far too small in magnitude compared to I! This can be shown as follows. Let us calculate the values of mr^2 for the maximum value of m: say $m = 0.26$ Kg. As the disc-pulley has a radius of 1.5 cm, we get:

$$mr^2 = 0.26 \times 0.015^2 = 5.85 \times 10^{-5} \text{ kgm}^2$$

The expected value of I is:

$$12 \times 10^{-3} \text{ kgm}^2$$

We find that I is some 200 times greater than mr^2 and as such, mr^2 may be ignored altogether. Deleting it from Equation (27) we get:

$$mgr = I\alpha \qquad ....(28)$$

This is what we get by considering the *dynamics* part of the motion of our system. We shall next investigate the use of *kinematics* to find α.

Determining the Angular Acceleration α

Consider Equation (9):

$$\theta = \frac{(\omega^2 - \omega_o^2)}{2\alpha}$$

Or

$$\alpha = \frac{(\omega^2 - \omega_o^2)}{2\theta} \qquad ....(29)$$

The photogate timer can be used to measure the angular velocity of rotation of the system of discs. An interesting method is to cut out a 1-radian sector of angular distance from a cardboard or stiff paper, and place it in between the two discs. It should protrude beyond the discs by couple of centimeters. The photogate may be positioned in such a way that this 1-radian sector would pass through its window. This is shown in Fig (2). The timer is set in the *gate* mode. As the 1-radian sector passes through the photogate, the time t_{pass} is recorded. Now since $\Delta\theta = 1$ rad:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{1}{\Delta t} = \frac{1}{t_{pass}} \qquad ....(30)$$

Thus the reciprocal of t_{pass} is the angular velocity ω of the system of discs in radians per second. One can measure several of these velocities and calculate the angular acceleration α. One may choose a number of different values of m and determine α for each. A graph of mgr against α should yield a straight line of slope I.

Apparatus Required

(1) System of discs, mounted on a vertical axis, on frictionless spindle,

(2) Electronic timer

(3) Photogate

(4) A massless and frictionless pulley with clamp to be mounted at the end of the table

(5) Digital balance, capable of measuring up to one-hundredth of a gram,

(6) Mass-holder, unstretchable cord, set of masses, 6" ruler, a 1-radian sector of cardboard.

Procedure

Following are some guidelines for performing the experiment.

1) Set up the apparatus making sure that, to the best of your visual judgement, the cord is horizontal.

2) Wind the cord rather **tightly** on the top most pulley (radius = 1.5 cm) either in clockwise or counterclockwise direction. But always stick to the one you chose, for all trials. The cord should be wound until m is nearly as high as it will be without hitting the clamp that holds the pulley at the end of the table.

3) The starting position of the 1-radian sector should be such that it is just outside the window of the photogate timer in such a way that as it begins to rotate, it moves *away* from the window. Fig (2) shows the arrangement for the rotation of the discs in the clockwise direction (the cord having been wound in the counterclockwise direction).

4) Carry out couple of trial runs to make sure that during unwinding, m keeps moving downward. If the cord is not long enough m may start moving back (upward) before the number of necessary rotations have been completed, thereby decelerating the discs. One way of avoiding this is to let the cord be long enough so that when fully unwound, m would ***hit*** the floor. The sound of hitting the floor will tell you that m is not going down any more. Rest assured, however, the discs will complete more than the required number of rotations, *before* this happens.

5) Set the electronic timer in ***Rotational Inertia*****,** (or the ***gate)*** mode and select 6 memories. In this mode, the timer will automatically record six values of time, one each for six consecutive rotations of the discs. The time will be measured up to one-tenth of a millisecond.

6) Select 12 values of m : $40g$, $60g$,.............$260g$.

7) For each of the 12 prescribed masses do the following:

(i) find the total mass (m *plus* the mass of the mass-holder) using the digital balance.

(ii) wind the cord as per step (2) and hold the discs at rest. The photogate should be positioned as described in step (3) and shown in Fig (2). Press *clear* on the timer.

(iii) release the system of discs gently from rest. When the discs have completed 6 rotations, you may stop them manually. The timer will have, in the meantime, recorded the six corresponding values of t_{pass}, and stored them in memory #s 1 through 6.

(iv) ignoring the first value of t_{pass}, read the next five values by recalling memory #s 2 through 6. Record these values of t_{pass} in your data table as t_1, t_2, t_3, t_4, and t_5.

8) The experiment ends. Arrange all apparatus neatly on the table.

Calculations and Graph

1) Calculate the 5 angular velocities:

$$\omega_1 = (1/t_1), \quad \omega_2 = (1/t_2), \quad \omega_3 = (1/t_3), \quad \omega_4 = (1/t_4), \quad \omega_5 = (1/t_5)$$

2) Calculate 4 values of angular accelerations as:

$$\alpha_1 = \frac{1}{4\pi}(\omega_2^2 - \omega_1^2) \quad \alpha_2 = \frac{1}{4\pi}(\omega_3^2 - \omega_2^2) \quad .... \quad \alpha_4 = \frac{1}{4\pi}(\omega_5^2 - \omega_4^2)$$

(3) Find average angular acceleration α_{av}.

4) Repeat steps 1 through 3 for the next 11 values of m, to get 12 values of α_{av}.

5) For each of the 12 values of m calculate mgr; where $g = 9.80$ m/s^2 and $r = 0.015$ m .

6) Plot values of mgr on y-axis and the values of α_{av} on the x-axis; draw a best fit straight line and find its slope. This is I. Compare with its expected value of 11.98 x 10^{-3} kg m^2 and find percent error.

Note (1): Some new ones may have different values of I. Refer to the instructor.

Note (2): All calculations can be done on computer. The format developed, can be loaded in all computers in the lab. Tables #s 3, 4, and 5 will become redundant.

Conclusions and Discussions

Write your conclusions from the experiment and discuss them.

What Did You Learn in this Experiment?.

A hearty and thoughtful account of what you learned in this experiment by way of the principle and the techniques of experimentation, should be given

Data & Data Tables

Name.................................... **Date**..............................

Instructor.............................. **Lab Section**....................

Partner.......................... **Table #**..........................

(a) Radius of the aluminium pulley: 1.5 cm = 0.015 m

(b) Acceleration due to gravity: 9.80 (m/s^2)

To find the total mass m (selected mass *plus* the mass of the mass holder) please use the digital balance

.

Table 1: Data for Determining the Rotational Inertia of The Given System of Discs

# (1)	Prescribed mass (g) (2)	Total mass m (g) (3)	Total mass m (kg) (4)	t_1 (sec) (5)	t_2 (sec) (6)	t_3 (sec) (7)	t_4 (sec) (8)	t_5 (sec) (9)
1	40							
2	60							
3	80							
4	100							
5	120							
6	140							
7	160							
8	180							
9	200							
10	220							
11	240							
12	260							

Additional information or data (if any):

Spare data sheet, in case..........

(a) Radius of the aluminium pulley: 1.5 cm = 0.015 m

(b) Acceleration due to gravity: 9.80 (m/s^2)

To find the total mass m (selected mass *plus* the mass of the mass holder) please use the digital balance

.

Table 2: Data for Determining the Rotational Inertia of The Given System of Discs

# (1)	Prescribed mass (g) (2)	Total mass m (g) (3)	Total mass m (kg) (4)	t_1 (sec) (5)	t_2 (sec) (6)	t_3 (sec) (7)	t_4 (sec) (8)	t_5 (sec) (9)
1	40							
2	60							
3	80							
4	100							
5	120							
6	140							
7	160							
8	180							
9	200							
10	220							
11	240							
12	260							

Additional information or data (if any):

Table 3: Calculating Angular Velocities for Each of the Five Rotations

Trial #	$\omega_1 = 1/t_1$ (rad/sec)	$\omega_2 = 1/t_2$ (rad/sec)	$\omega_3 = 1/t_3$ (rad/sec)	$\omega_4 = 1/t_4$ (rad/sec)	$\omega_5 = 1/t_5$ rad/sec)
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					

Table 4: Calculating Average Angular Acceleration for Each Trial

Trial # (1)	$\alpha_1 = \frac{\omega_2^2 - \omega_1^2}{4\pi}$ (rad/sec^2) (2)	$\alpha_2 = \frac{\omega_3^2 - \omega_2^2}{4\pi}$ (rad/sec^2) (3)	$\alpha_3 = \frac{\omega_4^2 - \omega_3^2}{4\pi}$ (rad/sec^2) (4)	$\alpha_4 = \frac{\omega_5^2 - \omega_4^2}{4\pi}$ (rad/sec^2) (5)	$\alpha_{av} = \frac{\Sigma\alpha_n}{4}$ (rad/sec^2) (6)
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					

The product of g and r is:

$$gr = 9.80 \times 0.015 = 1.470 \quad (m^2/s^2)$$

Table 5: Calculating Torques & Preparing (x,y) Data Table

Trial #	m from column 4 table 1 (kg)	gr (m^2/s^2)	mgr (Nm)	α_{av} from last column table 4 (rad/sec^2)
1		0.147		
2		0.147		
3		0.147		
4		0.147		
5		0.147		
6		0.147		
7		0.147		
8		0.147		
9		0.147		
10		0.147		
11		0.147		
12		0.147		
			y-axis	x-axis