Table 1: Eigenfunctions of Simple Harmonic Oscillator

	"Hermite" Family of Equations	"Z" Family of Equations
The Generating Function	$T(t,\zeta) = e^{\zeta^2-(t-\zeta)^2} = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)(t^n)H_n(\zeta)$	$S(s,\zeta) = e^{\frac{\zeta^2}{2}-\left(\frac{1}{2}\right)(s-\sqrt{2}\zeta)^2} = \sqrt{\frac{\sqrt{\pi}}{n!}}(s^n)Z_n(\zeta)$
The Polynomial	$H_n(\zeta) = (-1)^n(e^{\zeta^2})\frac{d^n}{d\zeta^n}(e^{-\zeta^2})$	$Z_n(\zeta) = \frac{(-1^n)}{\sqrt{2^n n!\sqrt{\pi}}}(e^{\zeta^2/2})\frac{d^n}{d\zeta^n}(e^{-\zeta^2})$
The First Recurrence Relation	$H_{n+1}(\zeta) - 2\zeta H_n(\zeta) + 2nH_{n-1}(\zeta) = 0$	$\sqrt{n+1}\,Z_{n+1}(\zeta) - \sqrt{2}\,\zeta Z_n(\zeta) + \sqrt{n}\,Z_{n-1}(\zeta) = 0$
The Second Recurrence Relation		$\sqrt{(n+1)(n+2)}Z_{n+2}(\zeta) - (2\zeta^2 - 2n - 1)Z_n(\zeta)$ $+ \sqrt{n(n-1)}Z_{n-2}(\zeta) = 0$
The Differential Expression	$H_n'(\zeta) = 2n\,H_{n-1}(\zeta)$	$Z_n'(\zeta) = \sqrt{2n}\,Z_{n-1}(\zeta) - \zeta\,Z_n(\zeta)$
The Differential Equation	$H_n''(\zeta) - 2\zeta\,H_n'(\zeta) + 2nH_n(\zeta) = 0$	$Z_n''(\zeta) + [(2n+1) - \zeta^2]Z_n(\zeta) = 0$
Orthogonality $(m \neq n)$	$\int_{-\infty}^{\infty} H_n(\zeta)H_m(\zeta)\,d\zeta = 0$	$\int_{-\infty}^{\infty} Z_n(\zeta)Z_m(\zeta)\,d\zeta = \delta_{mn}$
Normality $(m = n)$	$\int_{-\infty}^{\infty} H_n(\zeta)H_m(\zeta)\,d\zeta = \frac{1}{\sqrt{\alpha\sqrt{\pi}(2^n)(n!)}}$	
The Explicit Representation	$H_n(\zeta) = (\sqrt{2^n})(n!)\left[\sum_{k=0}^{n/2}\frac{(-1)^k(\sqrt{2}\zeta)^{n-2k}}{(2^k)(k!)(n-2k)!}\right]$	$Z_n(\zeta) = \left(\frac{n!}{\sqrt{\pi}}\right)^{1/2}\left[\sum_{k=0}^{n/2}\frac{(-1)^k(\sqrt{2}\zeta)^{n-2k}}{(2^k)(k!)(n-2k)!}\right]e^{-\zeta^2/2}$

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The zeroth Function	$H_o(\zeta) = 1$	$Z_o(\zeta) = \left(\frac{1}{\sqrt{0!\sqrt{\pi}}}\right) e^{-\zeta^2/2}$
The First Function	$H_1(\zeta) = 2\zeta$	$Z_1(\zeta) = \left(\frac{1}{\sqrt{1!\sqrt{\pi}}}\right)(\sqrt{2}\zeta)\, e^{-\zeta^2/2}$
The Second Function	$H_2(\zeta) = 4\zeta^2 - 2$	$Z_2(\zeta) = \left(\frac{1}{\sqrt{2!\sqrt{\pi}}}\right)[(\sqrt{2}\zeta)^2 - 1]\, e^{-\zeta^2/2}$
The Third Function	$H_3(\zeta) = 8\zeta^2 - 12\zeta$	$Z_3(\zeta) = \left(\frac{1}{\sqrt{3!\sqrt{\pi}}}\right)[(\sqrt{2}\zeta)^3 - 3(\sqrt{2}\zeta)]\, e^{-\zeta^2/2}$
The Fourth Function	$H_4(\zeta) = 16\zeta^4 - 48\zeta^2 + 12$	$Z_4(\zeta) = \left(\frac{1}{\sqrt{4!\sqrt{\pi}}}\right)[(\sqrt{2}\zeta)^4 - 6(\sqrt{2}\zeta)^2 + 3]\, e^{-\zeta^2/2}$
The Fifth Function	$H_5(\zeta) = 32\zeta^5 - 160\zeta^3 + 120\zeta$	$Z_5(\zeta) = \left(\frac{1}{\sqrt{5!\sqrt{\pi}}}\right)[(\sqrt{2}\zeta)^5 - 10(\sqrt{2}\zeta)^3 + 15(\sqrt{2}\zeta)]\, e^{-\zeta^2/2}$

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Application to The Classical Linear Harmonic Oscillator

The Hamiltonian Function H of a simple harmonic oscillator in an elastic force field is:

$$H = \frac{p_x^2}{2m} + \frac{1}{2}Kx^2$$

The Hamiltonian Operator

$$\vec{H} = -\frac{\bar{h}^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}Kx^2$$

Assuming the existence of $\psi(x)$ such that:

$$\vec{H}\psi(x) = E\psi(x) \qquad \text{..........(1)}$$

we get:

$$-\frac{\bar{h}^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{1}{2}Kx^2\psi(x) = E\psi(x)$$

For:

$$\alpha^2 = \frac{\bar{h}}{m\omega}, \qquad \omega = \sqrt{\frac{K}{m}}, \qquad \text{and} \qquad \lambda = \frac{2E}{\bar{h}\omega} \qquad \text{..........(2)}$$

and

$$x = \alpha\zeta \qquad dx = \alpha(d\zeta) \qquad \frac{d}{dx} = \frac{1}{\alpha}\frac{d}{d\zeta} \qquad \frac{d^2}{dx^2} = \frac{1}{\alpha^2}\frac{d^2}{d\zeta^2} \qquad \psi(x) = \psi(\alpha\zeta) = \phi(\zeta) \qquad \text{..........(3)}$$

$$\phi''(\zeta) + (\lambda - \zeta^2)\phi(\zeta) = 0 \qquad \text{..........(4)}$$

Eqn (4) is the Schroedinger equation for the oscillating particle.

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Necessary transformation	$\psi(x) = \phi(\zeta) = e^{-\zeta^2/2}\chi(\zeta)$	$\psi(x) = \psi(\alpha\zeta) = Z(\zeta)$
Schroedinger Equation reduces to:	$\frac{d^2}{d\zeta^2}\chi(\zeta) - 2\zeta\frac{d}{d\zeta}\chi(\zeta) + (\lambda - 1)\chi(\zeta) = 0$	$\frac{d^2}{d\zeta^2}\psi(\zeta) + [\lambda - \zeta^2]\psi(\zeta) = 0$
Setting and	$\lambda = (2n+1)$ $\chi_n(\zeta) = H_n(\zeta)$	$\lambda = (2n+1)$ $\psi_n(x) = Z_n(\zeta)$
Schroedinger Equation reduces to:	$H_n''(\zeta) - 2\zeta H_n'(\zeta) + 2nH_n(\zeta) = 0$	$Z_n''(\zeta) + (2n+1-\zeta^2)Z_n(\zeta) = 0$
Eigenvalues	$E_n = (n+\frac{1}{2})(\bar{h}\omega)$	$E_n = (n+\frac{1}{2})(\bar{h}\omega)$
Eigenfunctions	$\psi_n(x) = [N_n e^{-\zeta^2/2}]\, H_n(\zeta)$	$\psi_n(x) = \left(\frac{1}{\sqrt{\alpha}}\right)Z_n(\zeta)$
Attachment for Eigenfunctions	$e^{-\zeta^2/2}$	$(1/\sqrt{\alpha})$

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Normalizing Factors for the First Five Eigenfunctions are:		
Normalizing Factor	$\dfrac{1}{\sqrt{\alpha\sqrt{\pi}(2^n)(n!)}}$	***NONE***
$n = 0$	$N_o = \dfrac{1}{\sqrt{\alpha\sqrt{\pi}}}$	***NONE***
$n = 1$	$N_1 = \dfrac{1}{\sqrt{2\alpha\sqrt{\pi}}}$	***NONE***
$n = 2$	$N_2 = \dfrac{1}{2\sqrt{2\alpha\sqrt{\pi}}}$	***NONE***
$n = 3$	$N_3 = \dfrac{1}{4\sqrt{3\alpha\sqrt{\pi}}}$	***NONE***
$n = 4$	$N_4 = \dfrac{1}{8\sqrt{6\alpha\sqrt{\pi}}}$	***NONE***
$n = 5$	$N_5 = \dfrac{1}{16\sqrt{15\alpha\sqrt{\pi}}}$	***NONE***

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	"Hermite" Family of Equations	"Z" Family of Equations
Expectation Integral	$(\psi_m(x), x\,\psi_n(x)) = \int_{-\infty}^{\infty} \psi_m(x)\,\psi_n(x) x\,dx$ $= \alpha^2 N_m N_n \int_{-\infty}^{\infty} H_n(\zeta)\,H_m(\zeta)\,e^{-\zeta^2}\zeta\,d\zeta$	$(\psi_m(x), x\,\psi_n(x)) = \int_{-\infty}^{\infty} \psi_m(x)\,\psi_n(x) x\,dx$ $= \alpha \int_{-\infty}^{\infty} Z_n(\zeta)\,Z_m(\zeta)\zeta\,d\zeta$
Expectation Value for $m = n+1$	$\alpha^2 N_m N_n \int_{-\infty}^{\infty} H_n(\zeta) H_m(\zeta)\,e^{-\zeta^2}\zeta\,d\zeta = \alpha\sqrt{\frac{n+1}{2}}$	$\alpha \int_{-\infty}^{\infty} Z_n(\zeta)\,Z_m(\zeta)\zeta\,d\zeta = \alpha\sqrt{\frac{n+1}{2}}$
Expectation Value for $m = n-1$	$\alpha^2 N_m N_n \int_{-\infty}^{\infty} H_n(\zeta) H_m(\zeta)\,e^{-\zeta^2}\zeta\,d\zeta = \alpha\sqrt{\frac{n}{2}}$	$\alpha \int_{-\infty}^{\infty} Z_n(\zeta)\,Z_m(\zeta)\zeta\,d\zeta = \alpha\sqrt{\frac{n}{2}}$