Z-Polynomials for Orthonormal Eigenfunctions of Simple Harmonic Oscillators

(Dr. Zafar A. Ismail, Daemen College, Amherst, New York)

The Hermite Polynomials

The energy eigenfunctions of a particle undergoing simple harmonic oscillations in an elastic force field, are typically described in terms of Hermite Polynomials, $H_n(\zeta)$. Hermite and other polynomials were developed by mathematicians long before the advent of Quantum Mechanics. Because of their comprehensive formalism, they are immensely useful and physicists find them irresistible. Yet, these are not tailor-made for the special needs of a given problem in physics.

Hermite Polynomials are best suited for the eigenfunctions of a simple harmonic oscillator. But as they were not designed with our oscillator in mind, we find following shortcomings:

(i) The eigenfunctions of hermite Polynomials are ***not normalized***. A factor C_n is left out which must be computed separately for each eigenfunction.

(ii) The eigenfunctions $\psi_n(x)$, when written out in full, are found to contain mutually cancelling factors in $e^{-\zeta^2}$, as shown below: It should be noted that the differentiation process will yield yet another $e^{-\zeta^2}$

$$\psi_n(x) = \left(\frac{(-1)^n}{\sqrt{2^n n!\, \alpha\sqrt{\pi}}}\right) e^{-\zeta^2/2} e^{\zeta^2} \frac{d^n}{d\zeta^n}\left(e^{-\zeta^2}\right)$$

(iii) Recurrence formulae give us another Hermite polynomial and not another oscillator eigenfunction. Finding a new eigenfunction is therefore, an indirect and lengthy process.

The Z-Polynomials

Z-Polynomials are tailor-made for the energy eigenfunctions of the above mentioned simple harmonic oscillator. The functions are orthonormal, by nature and do not have the disadvantages mentioned above. It was found possible to “absorb” all the factors needed for normalization into a new generating function and thereby evolve a new set of polynomials, the ***Z-Polynomials.***

Z-Polynomials are complete with generating function, recurrence formulae, differential equation, explicit representation, and everything else that Hermite Polynomials have. They produce orthogonal ***and*** normalized set of eigenfunctions of the harmonic oscillator directly. They also produce correct expectation values.

Table (1) lists all major properties and equations of the two families of polynomials, and thereby provides a direct comparison. The essentials of weaving them in the treatment of the simple harmonic oscillator have also been shown.

Conclusion

Z-Polynomials are better suited for finding the energy eigenfunctions and eigenvalues of a one-dimensional simple harmonic oscillator, as it oscillates in an elastic force field.

(e-mail: zismail@daemen.edu)